# Classical Electrodynamical Derivation of the Radiation Damping Force

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A covariant expression for the instantaneous radiation damping force acting on an accelerated charged particle is derived within the frame of classical electrodynamics. The radiation pressure of the wave emitted by the charge is averaged on a sphere of radius $R$ to obtain the net force due to the photon momentum recoil, and the limit is taken when $R$ tends to zero, assuming no internal structure of the particle. The relativistic Doppler effects break the symmetry of the instantaneous rest frame dipole radiation pattern, and the Abraham-Becker damping force is obtained.

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The derivation of an exact expression for the radiation damping force has long been an outstanding problem of classical electrodynamics [1–7,11–13]. As a charged particle is submitted to an external force and accelerated, it radiates electromagnetic energy [8–13]. The recoil momentum of the photons emitted during this process is equivalent to a reaction force corresponding to the self-interaction of the particle with its own electromagnetic field. A number of derivations of the radiation damping force have been given in the past. In the classic derivation given by Abraham and Lorentz [11,12], which relies on energy-momentum conservation, the self-electromagnetic-energy and momentum of a charged rigid sphere are derived for an accelerated motion. In its first-order approximation, this derivation yields the well-known Abraham-Lorentz force which depends on the second time derivative of the particle velocity. The main difficulties with this solution have been thoroughly discussed by Jackson [13] and Becker [11]. Another approach to the problem yields an integrodifferential equation of motion for a charged particle which includes radiation damping [13]; the solutions to this equation of motion predict acausal "preacceleration" on distances of the order of the classical radius of the electron, as discussed by Jackson [13]. Generally, both types of derivations are first performed in the instantaneous rest frame of the particle, and then generalized by finding the corresponding covariant expression of the radiation damping force.

The purpose of this Letter is to propose what we believe to be a simple alternate derivation of an expression of the instantaneous radiation damping force acting on an accelerated charged particle, while keeping a special emphasis on the physics of the electron self-interaction. This Letter is organized as follows. For the sake of simplicity, the expression of the damping force is first derived in an instantaneous frame where the particle velocity and acceleration are collinear. The expression obtained is then generalized by examining its covariance. To show the generality of this approach, we start the derivation by demonstrating that it is always possible to find such an instantaneous reference frame where the particle velocity and acceleration are aligned. We then proceed by deriving the expression of the radiation damping force in this particular frame. Because the particle velocity is not set to zero (by contrast with previous calculations done in the instantaneous rest frame of the charge), some essential features of radiation damping physics are preserved, and can be clearly exhibited. The derivation is performed by considering a spherical surface of radius $R$ centered around the particle at the retarded time when the velocity and acceleration are collinear, and deriving the expression of the radiation pressure of the electromagnetic wave radiated. The radiation pressure is then integrated over the sphere to obtain the net force due to the photon momentum recoil, and the limit is taken when $R$ tends to zero, assuming no internal structure of the particle. When the radius is equal to zero, the expression derived corresponds to the instantaneous radiation damping force acting on the charged particle. A discussion of the underlying physics, and a comparison with previously derived expressions of the radiation damping force, are then given.

We now consider a Galilean frame $L$, where a charged particle moves along the world line

$$r_\nu(\tau) \equiv (\mathbf{r}, ict), \tag{1}$$

where $\tau$ is the particle proper time ($dt = \gamma\, d\tau$), and where the particle normalized velocity, normalized energy, and acceleration are defined as

$$\boldsymbol{\beta} = \frac{d\mathbf{r}}{c\,dt}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \dot{\boldsymbol{\beta}} = \frac{d\boldsymbol{\beta}}{dt}. \tag{2}$$

The four-velocity and four-acceleration are then defined as [3,11]

$$u_\nu = \frac{dr_\nu}{d\tau} = \gamma(\boldsymbol{\beta}, i), \tag{3}$$

$$a_\nu = \frac{du_\nu}{d\tau} = \gamma^2[\dot{\boldsymbol{\beta}} + \gamma^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}), i\gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})]. \tag{4}$$

The expression for the normalized velocity and acceleration in any other Galilean frame $L'$ moving with respect to $L$ with normalized velocity $\mathbf{w}$ can be derived by applying the Lorentz transform to Eqs. (3) and (4) to obtain the following results [14]:

$$\boldsymbol{\beta}' = \frac{\alpha\boldsymbol{\beta} + \mathbf{w}[(1-\alpha)(\boldsymbol{\beta}\cdot\mathbf{w}/w^2) - 1]}{1 - \boldsymbol{\beta}\cdot\mathbf{w}}, \qquad (5)$$

$$\dot{\boldsymbol{\beta}}' = \frac{\alpha}{(1-\boldsymbol{\beta}\cdot\mathbf{w})^2}\left[\alpha\dot{\boldsymbol{\beta}} + \mathbf{w}(1-\alpha)\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{w}}{w^2} + \boldsymbol{\beta}'(\dot{\boldsymbol{\beta}}\cdot\mathbf{w})\right]. \qquad (6)$$

Here the primed quantities refer to the new reference frame, and the parameter $\alpha$ is defined as

$$\alpha = \sqrt{1 - w^2}. \qquad (7)$$

The third term on the right-hand side of the expression for the acceleration given in Eq. (6) is already proportional to the normalized particle velocity as measured in $L'$. It can then be shown after some straightforward vector calculus that if we choose

$$\mathbf{w} = \boldsymbol{\beta} - g\dot{\boldsymbol{\beta}}, \qquad (8)$$

where $g$ is an arbitrary parameter which has the dimension of time (provided that $g$ is such that $|\mathbf{w}| < 1$), we end up with

$$\dot{\boldsymbol{\beta}}' = \frac{\boldsymbol{\beta}'}{g}\frac{\alpha^3}{(1-\boldsymbol{\beta}\cdot\mathbf{w})^2}. \qquad (9)$$

Equation (9) can also be written explicitly in terms of the particle acceleration and velocity in $L$ as

$$\dot{\boldsymbol{\beta}}' = \frac{\boldsymbol{\beta}'}{g}\frac{(1 - \beta^2 - g^2\dot{\beta}^2 + 2g\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^{3/2}}{(1 - \beta^2 + g\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2}. \qquad (10)$$

Therefore, it is always possible to transform to an instantaneous Galilean frame where the velocity and acceleration are collinear.

The geometry of the radiation process as observed in a reference frame where the velocity and acceleration are collinear is illustrated in Fig. 1. For convenience, we will call this frame $L$ in the remainder of the text. For a single point charge $e$, describing a trajectory $\mathbf{r}(t)$, the electric field at $x_\mu$ is obtained by deriving the Liénard-Wiechert four-vector potential. In mksa units, we have for the radiative field [3–5,11,13]

$$\mathbf{E}(\mathbf{x},t) = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]^{-}, \qquad (11)$$

where the quantities in the square bracket are evaluated at the retarded time $t^-$ such that

$$c(t - t^-) = R(t^-) = |\mathbf{x} - \mathbf{r}(t^-)|. \qquad (12)$$

Here, $\mathbf{n}$ is the unit vector in the direction of observation, as shown in Fig. 1, and the other quantities are measured in a reference frame $L$, where the acceleration and velocity are collinear. The instantaneous energy flux is given in terms of the Poynting vector, defined as

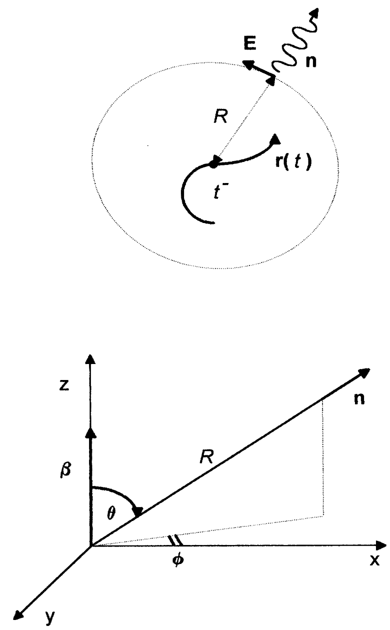


FIG. 1. Geometry of the radiation process.

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \mathbf{n}\frac{E^2}{\mu_0 c}. \qquad (13)$$

The total radiation pressure force applied to a sphere of radius $R$, corresponding to the momentum recoil of the photons emitted by the particle at $t^-$, is then given by

$$\int\int \frac{\mathbf{S}}{c} R^2\, d\Omega. \qquad (14)$$

We propose that the instantaneous radiation damping force acting on the charged particle as it radiates is given by

$$\mathbf{F} = -\lim_{R\to 0}\left[\int\int \frac{\mathbf{S}}{c} R^2\, d\Omega\right], \qquad (15)$$

according to the principle of action and reaction. Here, we have made the implicit assumption that the charged particle has no internal structure. Thus, we have

$$\mathbf{F} = -\lim_{R\to 0}\left[\int\int \mathbf{n}\varepsilon_0 E^2 R^2\, d\Omega\right], \qquad (16)$$

which reduces to

$$\mathbf{F}(t) = -\frac{\mu_0 e^2}{16\pi^2}\int\int \mathbf{n}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]^2\Bigg|_{t^-=t} d\Omega \qquad (17)$$

At this point, it is important to note that, as the sphere radius tends to zero, the retarded time tends to the instantaneous interaction time. Following Sommerfeld [4], we now express the force as a function of the particle proper time:

$$\mathbf{F}(\tau) = -\frac{\mu_0 e^2}{16\pi^2} \int\int \mathbf{n}\left[\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3}\right]^2 \frac{dt}{d\tau}\, d\Omega\,, \tag{18}$$

which yields

$$\mathbf{F}(\tau) = -\frac{\mu_0 e^2}{16\pi^2} \int\int \mathbf{n}\frac{\{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]\}^2}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^5}\, d\Omega\,. \tag{19}$$

To evaluate the integral in Eq. (19), we use the collinearity of the velocity and acceleration in $L$; therefore, we have

$$\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = \mathbf{0}, \quad \{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]\}^2 = \dot{\beta}^2 - (\mathbf{n} \cdot \dot{\boldsymbol{\beta}})^2. \tag{20}$$

As shown in Fig. 1, we have chosen the axis of $L$ such that we have

$$\boldsymbol{\beta} = \hat{\mathbf{z}}\beta\,,$$
$$\dot{\boldsymbol{\beta}} = \hat{\mathbf{z}}\dot{\beta}\,,$$
$$\mathbf{n} = \hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}\cos\theta\,.$$

With this, Eq. (20) yields

$$\{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]\}^2 = \dot{\beta}^2 \sin^2\theta\,. \tag{21}$$

Therefore, the expression for the instantaneous radiation damping force in $L$ reduces to

$$\mathbf{F}(\tau) = -\frac{\mu_0}{16\pi^2} e^2\dot{\beta}^2 \int_0^{2\pi} d\phi \int_0^{\pi} \mathbf{n}\,\frac{\sin^3\theta}{(1 - \beta\cos\theta)^5}\, d\theta\,. \tag{22}$$

The integrals corresponding to the $x$ and $y$ components average to zero over $\phi$, and we are left with

$$\mathbf{F}(\tau) = -\hat{\mathbf{z}}\frac{\mu_0}{8\pi} e^2\dot{\beta}^2 \int_0^{\pi} \frac{\sin^3\theta\cos\theta}{(1 - \beta\cos\theta)^5}\, d\theta\,. \tag{23}$$

This integral can be performed by using the new variable $x = \cos\theta$; we then have to integrate the ratio of two polynomials,

$$\int_0^{\pi} \frac{\sin^3\theta\cos\theta}{(1 - \beta\cos\theta)^5}\, d\theta = \int_{-1}^{+1} \frac{x(1 - x^2)}{(1 - \beta x)^5}\, dx = \frac{4}{3}\frac{\beta}{(1 + \beta)^3(1 - \beta)^3}\,. \tag{24}$$

After some straightforward algebra, we end up with the sought-after expression for the radiation damping force in $L$,

$$\mathbf{F}(\tau) = -\boldsymbol{\beta}\frac{\mu_0}{6\pi} e^2\dot{\beta}^2\gamma^6. \tag{25}$$

The self-interaction nature of the radiation damping force is evident, as the expression derived scales with the square of the particle charge. We also recover the well-known quadratic scaling with the acceleration. However, the most interesting feature of the expression given in Eq. (25) is the linearity of the force with respect to the particle velocity. Within the model presented here, this indicates that the origin of the damping force is essentially a relativistic effect. If we first consider the instantaneous rest frame of the particle ($\beta = 0$), as illustrated in Fig. 2 (top), we see that the damping force vanishes. This is due to the symmetry of the dipole radiation pattern in this particular frame: Although electromagnetic energy is radiated by the particle, there is no net recoil force because, for each photon radiated in a given direction of space, there is a photon with the same momentum radiated in the opposite direction. In any other Galilean frame, the relativistic Doppler effect breaks this symmetry (see Fig. 2, bottom): The photons radiated in the forward direction are now blueshifted and carry more momentum than their backscattered counterparts which are redshifted, resulting in a net radiation damping force opposite to the direction of motion. This is reflected by the force given in Eq. (25). In addition, this force has a nonzero value for a particle submitted to a constant acceleration, as opposed to the Abraham-Lorentz force [11,12] which depends on the second time derivative of the particle velocity [12,13].

The instantaneous power radiated can be obtained in a similar way by integrating the Poynting vector flux over all solid angles, with the well-known result that

$$\frac{dE}{dt}(\tau) = -\frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\dot{\beta}^2\,. \tag{26}$$

At this point, if we note that in $L$, where the velocity and acceleration are collinear, we have the formal identity

$$\gamma^6\dot{\beta}^2 = a_\nu a^\nu, \tag{27}$$

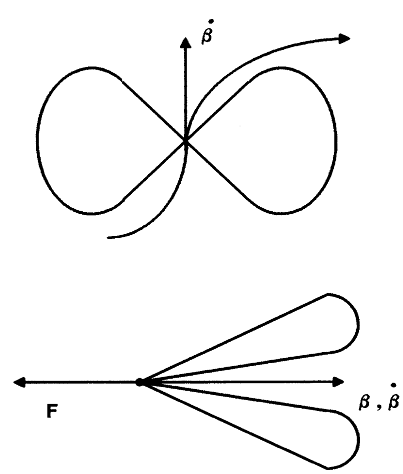


FIG. 2. Radiation patterns. Top: instantaneous rest frame. Bottom: instantaneous collinear frame.

the covariant generalization of Eqs. (25) and (26) becomes quite evident. Following Becker [11], we introduce the particle four-vector energy-momentum defined as

$$p_\nu = (\mathbf{p}, iE/c). \tag{28}$$

The covariant form of the energy-momentum transfer equation is then

$$\frac{dp_\nu}{d\tau} = -\frac{\mu_0 e^2}{6\pi}(a_\mu a^\mu)u_\nu. \tag{29}$$

The corresponding explicit expressions for the radiation damping force and radiated power are

$$\mathbf{F}(\tau) = \frac{d\mathbf{p}}{d\tau}\frac{d\tau}{dt} = -\frac{\mu_0 e^2}{6\pi}\gamma^6[\dot{\beta}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2]\boldsymbol{\beta}, \tag{30}$$

$$\frac{dE}{dt}(\tau) = c\frac{dp_0}{d\tau}\frac{d\tau}{dt} = -\frac{e^2}{6\pi\varepsilon_0 c}\gamma^6[\dot{\beta}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2], \tag{31}$$

respectively. Here, we have used the identity [11]

$$a_\nu a^\nu = \gamma^6[\dot{\beta}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2]. \tag{32}$$

Equation (30) corresponds to the expression of the radiation damping force derived by Abraham and Becker [11]; Eq. (31) is the well-known Liénard formula [13].

We now need to verify that energy conservation is satisfied. Again, to clearly show the physics involved, we use the expression for the instantaneous power as given by Pauli [3]:

$$\frac{dE}{dt} = -\frac{e^2}{6\pi\varepsilon_0 c}\gamma^4[\dot{\beta}^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2], \tag{33}$$

and we consider the reference frame $L$. The first term in Eq. (33) is independent of the velocity and corresponds to the dipolar emission in the instantaneous rest frame of the particle, which does not yield any damping force because of its symmetry. The second term contains the relativistic Doppler asymmetry and corresponds to the work of the damping force; therefore, in $L$, the energy conservation equation can be written

$$\left(\frac{dE}{dt} + \frac{e^2}{6\pi\varepsilon_0 c}\gamma^4\dot{\beta}^2\right) - \mathbf{F}\cdot c\boldsymbol{\beta} = 0. \tag{34}$$

In conclusion, we have proposed a simple alternate derivation of a covariant expression of the Abraham-Becker radiation damping force. Namely, the radiation pressure of the electromagnetic wave radiated by the particle is averaged on a sphere of radius $R$ to obtain the net force due to the photon momentum recoil, and the limit is taken when $R$ tends to zero, assuming no internal structure of the particle. The derivation, performed in an instantaneous rest frame where the charged particle velocity and acceleration are collinear, is independent of the internal structure of the particle and shows that the radiation damping force is essentially a relativistic effect: The relativistic Doppler effect breaks the symmetry of the dipolar radiation pattern observed in the instantaneous rest frame of the accelerated charge, yielding a net photon momentum recoil force. The merit of this derivation is to show in a simple way the essential physical processes at the origin of the radiation damping force.

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