Intermittency in Fully Developed Turbulence: Log-Poisson Statistics and Generalized Scale Covariance

Bérengère Dubrulle

Centre National de la Recherche Scientifique, UPR 182, Commissariat à l'Energie Atomique, Direction des Sciences de la Matiere, Departement d'Astrophysique, Physique de Particules, Physique Nucleaire et d'Instrumentation Associee, Service d'Astrophysique, Centre d'Etudes de Saclay, F-91191 Gif sur Yvette, France, and Centre National de la Recherche Scientifique, URA 285, Observatoire Midi Pyrénées, 14 avenue Belin, F-31400 Toulouse, France

(Received 3 March 1994)

Some properties of a model of intermittency in fully developed turbulence due to She and Lévêque [Phys. Rev. Lett. **72**, 336 (1994)] are explored. The probability functions solution of the model is shown to be simply related to the log-Poisson statistics of local nondimensional energy dissipation. It is also shown that the intermittency obtained by She and Lévêque can be interpreted as the consequence of the scale covariance of the energy dissipation. Based on these observations, a new picture of turbulence is presented, in which scale covariance plays a central role.

PACS numbers: 47.27.–i

In 1941, Kolmogorov (hereafter K41) conjectured the existence of a universal state in fully developed turbulence, in which the velocity differences δv_l across a distance l have a simple scaling behavior when l is in the inertial range:

$$\langle \delta v_l^p \rangle \sim l^{\zeta_p}, \quad \zeta_p = \frac{p}{3}. \tag{1}$$

Experiments and numerical simulations do not, however, support this conjecture. There is growing evidence that ζ_p substantially deviates from the linear K41 law at large p, a phenomenon often referred to as intermittency corrections. Paradoxically, there is also an indication that fully developed turbulence is more universal than thought by Kolmogorov. Benzi *et al.* [1] showed that scaling properties of the velocity increments can be extended up to the dissipative range under the form

$$\langle \delta v_l^p \rangle \sim \langle \delta v_l^3 \rangle^{\zeta_p/\zeta_3}, \quad l \geq 5\eta, \tag{2}$$

where η is the Kolmogorov scale. This property, referred to as extended self-similarity, is observed even at moderate Reynolds number. Extended self-similarity allows significant improvement in experimental measurements of ζ_p by extending the range of scales over which self-similarity holds, thereby providing a clear and unambiguous evidence for intermittency.

Many models have been proposed to explain the intermittency in ζ_p. The most famous are the log-normal model (the distribution of the energy dissipation is log-normal) [2,3], and the multifractal model (the energy dissipation has a multifractal measure) [4]. More recently, She and Lévêque [5] (hereafter SL) have proposed a simple model, which leads to a prediction of ζ_p in excellent agreement (1%) with experimental results (see Table I of SL). It is

$$\zeta_p = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right]. \tag{3}$$

The model is based on three simple hypotheses:

(i) The Kolmogorov refined similarity hypothesis is used: $\langle \delta v_l^p \rangle \sim l^{p/3} \langle \epsilon_l^{p/3} \rangle$, where ϵ_l is the energy dissipation averaged over a ball of size l.

(ii) "Hidden symmetry": The moments of the energy dissipation obey a hierarchical structure, given by

$$\frac{\langle \epsilon_l^{p+1} \rangle}{\epsilon_l^{(\infty)} \langle \epsilon_l^p \rangle} = A_p \left(\frac{\langle \epsilon_l^p \rangle}{\epsilon_l^{(\infty)} \langle \epsilon_l^{p-1} \rangle} \right)^\beta, \; 0 < \beta < 1, \tag{4}$$

where A_p are geometrical constants and $\epsilon_l^{(\infty)} \equiv \lim_{p\to\infty} \langle \epsilon_l^{p+1} \rangle / \langle \epsilon_l^p \rangle$ is a quantity tracing the tail of the distribution of ϵ_l, corresponding to very intermittent structures. SL postulate that the hierarchy originates from some hidden symmetry of the Navier-Stokes equations. This hypothesis has not yet been checked on real turbulent data. However, using a closure model, Benzi [6] showed recently that the hierarchical structure also holds in the GOY shell model [7], the simplest local deterministic approximation to Navier-Stokes equations which retains energy conservation and scale invariance. We later show that the hidden symmetry can be interpreted as generalized scale covariance, and that β is characteristic of the intermittency of energy dissipation.

(iii) The quantity $\epsilon_l^{(\infty)}$ appearing in (4) is associated with the most intermittent dissipative structures and shows a divergent scale dependence: $\epsilon_l^{(\infty)} \sim l^{-\Delta}$ as $l \to 0$, where Δ is a parameter depending on the codimension of the dissipative structure. The divergence property is derived by SL from interactions between the filaments and the surrounding fluid, which gives $\Delta = 2/3$.

Hypotheses (i) through (iii) guarantee that δv_l obey the scaling

$$\langle \delta v_l^p \rangle \sim l^{(1-\Delta)p/3 + \Delta(1-\beta^{p/3})/(1-\beta)}. \tag{5}$$

The fit of structure functions in isotropic 3D turbulence (SL model) requires $\Delta = \beta = 2/3$, which is obtained if

0031-9007/94/73(7)/959(4)$06.00

the most intermittent structures are filaments (of codimension two). The success achieved by this simple model is remarkable, even though hypotheses (ii) and (iii) still require experimental support or theoretical confirmation. The model therefore deserves further investigation. In this paper, we show that it is characterized by certain remarkable properties which may open new insights in our understanding of fully developed turbulence.

Log-Poisson statistics.—We introduce the nondimensional energy dissipation π_l as

$$\pi_l = \frac{\epsilon_l}{\epsilon_l^{(\infty)}}. \tag{6}$$

It may then be checked that the moment hierarchy (ii) implies that

$$\langle \pi_l^p \rangle = \frac{B_p}{B_1^{(1-\beta^p)/(1-\beta)}} \langle \pi_l \rangle^{(1-\beta^p)/(1-\beta)}, \tag{7}$$

where B_p are geometrical constants. This relation between moments is satisfied by a whole family of probability distribution functions. The members $P_{\lambda(l)}(\pi_l)$ of this family can be expressed simply as a function of $Y = \log \pi_l / \log \beta$ according to

$$P_{\lambda(l)}(\pi_l) d\pi_l = F_{\lambda(l)}(Y) dY,$$
$$F_{\lambda(l)}(Y) = \int \text{Poisson}_\lambda(Z) G(Y - Z) dZ, \tag{8}$$

where Poisson_λ is the Poisson distribution of expectation λ and G is any probability density function. The geometrical constants B_p appearing in (7) are then simply related to the Laplace coefficients of G via

$$B_p = \int_0^{+\infty} e^{pY \ln \beta} G(Y) dY. \tag{9}$$

From the expression of the first moment, it can be seen that the parameter λ is proportional to $\ln\langle \pi_l \rangle$.

Reciprocally, given the set of coefficient B_p and the parameter β, one may find the function $P_{\lambda(l)}$ of the form (8) which satisfies (7) by suitable Laplace transform. Therefore, the family of probability density functions defined by (8) are the general solutions of the moment hierarchy. Note that when $\beta \to 1$, the Poisson distribution tends to the Dirac function, and the family of functions is simply given by $P_{\lambda(l)}(\pi_l) = G(\pi_l / \langle \pi_l \rangle) \langle \pi_l \rangle^{-1}$, which is the general form of purely hierarchical probability density functions (such that $\langle \pi_l^n \rangle \propto \langle \pi_l \rangle^n$).

The simplest solution of (7) is obtained for $G(Y) = \delta(Y)$. In that case, $B_p = 1$ for all p, and π_l has a log-Poisson statistics. We shall see later that the log-Poisson statistics [and all the members of the family (8)] are characterized by special scale-covariance properties. In addition, it is interesting to note that the Poisson distribution occurs generally in connection with discontinuous random processes [8] and is the natural limit of a wide class of statistical distributions involving "rare events" (e.g., maxima of sequence of normal variables or cycle of mapping) [9]. The best example is given by the binomial distribution, the distribution of successes in a sequence of n independent trials, each with a probability x of success. If the number of trials tends to infinity, and x stays finite, the binomial distribution approaches the normal distribution. However, if the probability x is very close to zero (rare event) and $\lambda = nx$ is finite, the binomial distribution is very closely approximated by a Poisson distribution of expectation λ.

In this context, it is of interest to point out that the log-Poisson distribution can be obtained from the random β model [10] by a suitable limiting process. Specifically, the random β model is a modification of the β model of Ref. [11] in which the space-filling factor for offsprings takes two possible values, β_1 and β_2, with probabilities x and $1 - x$. In the β model, the random cascade factor between scales $i + 1$ and i noted $W = \epsilon_{i+1}/\epsilon_i$ takes only two values, 0 with probability $1 - \beta$ and $1/\beta$ with probability β. Hence, in the random β model, W takes three values:

$$W = \begin{cases} 0 & \text{with probability } 1 - \beta_1 x - \beta_2(1-x), \\ 1/\beta_1 & \text{with probability } \beta_1 x, \\ 1/\beta_2 & \text{with probability } \beta_2(1-x). \end{cases} \tag{10}$$

In addition, the random β model has a free parameter $0 < \Gamma < 1$, the ratio of successive scales. It is easily seen that $\langle \epsilon_l \rangle \sim l^{\tau_p}$, with $\tau_p = \log\langle W^p \rangle / \log \Gamma$. Let us now consider a family of random β models in which $x \to 0$ and in which Γ, β_1, and β_2 are functions of x and two additional parameters $0 < \beta < 1$ and $0 < \Delta < 3$, given by

$$\Gamma = \exp\left[-\frac{x(1-\beta)}{\Delta \beta}\right]; \; \beta_1 = \beta^{-1}\Gamma^{\Delta}; \; \beta_2 = \Gamma^{\Delta}. \tag{11}$$

A straightforward calculation shows that, for $x \to 0$, $\tau_p \to -p\Delta + \Delta(1 - \beta^p)/(1 - \beta)$, so that π_l has the log-Poisson form (7). This computation explicitly shows that the β parameter is a measure of the intermittency of energy dissipation [see (11)] and that Δ is linked to the codimension of the dissipative structures.

Generalized scale covariance.—We now show that the moment hierarchy (4) can also be interpreted in terms of covariance by a generalized scale transformation. For this, we recast (4) in an equivalent form. We may first note that, under its present form, the SL model does not account for extended self-similarity. This is because it explicitly involves the resolution scale l. However, if we recast hypotheses (i) and (iii) in terms of a "generalized scale" $\xi(l) = \langle \langle \epsilon_l \rangle^{-1} \delta v_l^3 \rangle$, the SL model can become fully compatible with extended self-similarity. Hypotheses (i), (ii), and (iii) then should be recast as

$$\frac{\delta v_l^3}{\langle \delta v_l^3 \rangle} \overset{\text{scal}}{=} \frac{\epsilon_l}{\langle \epsilon_l \rangle} = \frac{\pi_l}{\langle \pi_l \rangle}, \tag{12}$$

where $\stackrel{\text{scal}}{=}$ means we have the same statistical properties as far as scaling is concerned,

$$\frac{\langle \pi_l^{p+1} \rangle}{\langle \pi_l^p \rangle} = A_p \left(\frac{\langle \pi_l^p \rangle}{\langle \pi_l^{p-1} \rangle} \right)^{\beta}, \tag{13}$$

and

$$\langle \pi_l \rangle = \frac{\langle \epsilon_l \rangle}{\epsilon_l^{(\infty)}} = \left(\frac{\langle \delta v_l^3 \rangle}{\eta \langle \epsilon_l \rangle} \right)^{\Delta}. \tag{14}$$

Note that (12) has been already checked on real turbulent data [6]. Combination of Eqs. (12), (13), and (14) guarantees that the moments of the velocity increments obey the relation

$$\langle \delta v_l^p \rangle \sim \langle \delta v_l^3 \rangle^{\zeta_p/\zeta_3}, \tag{15}$$

with

$$\frac{\zeta_p}{\zeta_3} = (1-\Delta)\frac{p}{3} + \frac{\Delta}{1-\beta}(1-\beta^{p/3}). \tag{16}$$

One recovers therefore both extended self-similarity, and the SL model for $\zeta_3 = 1$ and $\Delta = \beta = 2/3$. Note, however, that this model could possibly be used in situations where ζ_3 is not equal to unity. In such a case, only the *relative* scaling exponents should follow the SL formula. This has been recently checked on a class of shell models [12].

The property of extended self-similarity suggests that scaling properties in turbulence should not be investigated as a function of l, the resolution scale, but rather as a function of the generalized scale $\xi(l,\eta)$, which may be simply taken as $\eta\langle\pi_l\rangle$ thanks to (14). This is reminiscent of critical phenomena in finite size systems, in which all the scaling properties are considered as a function of a correlation length, which depends on the size of the system, and diverges at the critical point. Our equivalent formulation of (4) is inspired from this remark. Indeed, as in SL, we now consider the following weighted probability density function:

$$Q_p(\pi_l) = \frac{\pi_l^p P(\pi_l)}{\int \pi_l^p P(\pi_l) d\pi_l}, \tag{17}$$

where $P(\pi_l)$ is the probability density function of π_l. We may then define the p average of π_l as

$$\langle \pi_l^n \rangle_p \equiv \int \pi_l^n Q_p(\pi_l) d\pi_l = \frac{\langle \pi_l^{n+p} \rangle}{\langle \pi_l^p \rangle}, \tag{18}$$

where $\langle\,\rangle$ is the average done using $P(\pi_l)$. We now investigate the scaling properties of the Q_p with respect to the generalized scale $\xi_p(l) \sim \langle \pi_l \rangle_p$. Since in the inertial range all $\langle \pi_l \rangle_p$ are power laws in the same variable l, they satisfy

$$\langle \pi_l^n \rangle_p \propto \langle \pi_l \rangle_p^{\zeta(n,p)} \quad \text{for all } n, p. \tag{19}$$

Here, the $\zeta(n,p)$ are functions of n and p to be determined. By definition, $\zeta(1,p) = 1$ for all p. Note that this formulation explicitly considers the scaling in $\xi_p(l)$ instead of l and uses relative exponents, in the spirit of Eq. (15).

Straightforward manipulations on $\langle \pi_l^n \rangle_p$ using the definition of $Q_p(\pi_l)$ show that the $\zeta(n,p)$ are necessarily linked by a simple recursion relation:

$$\zeta(n+1,p-1) = \zeta(n,p)[\zeta(2,p-1)-1]+1. \tag{20}$$

This relation is valid for any "scaling system." However, the SL moment hierarchy amounts to assuming that the function $\zeta(p,n)$ is *independent of p*. Indeed, in this case, the solution of (20) can be obtained exactly as

$$\begin{aligned} \zeta(p,n) \equiv \zeta(n) &= \frac{1-\beta^n}{1-\beta}, \text{ with } \beta \equiv \zeta(2)-1, \\ &= n \quad \text{if } \zeta(2) = 2\ (\beta = 1). \end{aligned} \tag{21}$$

In the present case, this p independence can be interpreted as stemming from covariance by a scale transformation. Indeed, it may be checked that the general family (8) of solutions of the moment hierarchy are such that

$$\begin{aligned} P_{\lambda(l)\beta^p}(\pi_l)d\pi_l &= F_{\lambda(l)\beta^p}(Y)dY \\ &= B_p \frac{e^{Y \ln \beta^p} F_{\lambda(l)}(Y)}{\langle \pi_l^p \rangle} dY \\ &= B_p Q_{\lambda(l),p}(\pi_l) d\pi_l. \end{aligned} \tag{22}$$

Here, the notation $Q_{\lambda(l),p}$ was used to define the weighted function of $P_{\lambda(l)}$ of order p. If one assumes covariance by dilation of $\lambda(l) \propto \ln\langle \pi_l \rangle$, the weighted probability density function of order p must then have the same scaling properties as $P_{\lambda(l)}(\pi_l)$, since it is its transform by a dilation of $\lambda(l)$ by a factor β^p. This automatically ensures that the $\zeta(n,p)$ are p independent.

When p varies, $\alpha = \beta^p$ can take any value between 0 and 1. Assuming covariance by dilation of $\lambda(l)$ by any factor β^p amounts then to assuming covariance by dilation of $\lambda(l)$ by any α, $0 \le \alpha \le 1$. Note that this dilation transforms $\langle \pi_l \rangle = \xi(l)/\eta$ into $O_\alpha(\xi(l)/\eta) = (\xi_l/\eta)^\alpha$. The set of transformations O_α is a semigroup, which generalizes the usual scale dilation $\xi(l) \to \gamma\xi(l)$ by using a scale-dependent dilation parameter $\gamma(\xi(l))$. In contrast with the usual dilation group, however, the set of transformations O_α all leave one scale invariant (here the cutoff scale η). They can therefore be interpreted as the natural generalization of scale dilations in systems in which the cutoff scale is always relevant, exactly like Lorentz transformations are the generalization of Galilean transformations in systems with a constant upper velocity.

If the SL model is correct, we are faced with a totally new interpretation of intermittency in fully developed turbulence. Turbulence develops via scale-covariant interactions, as modeled by the Navier-Stokes equations.

This scale covariance extends from the injection scale to the dissipative scale. Its signature is the hierarchical structure of the various moments of energy dissipation (4), characterized by an anomalous exponent β. This parameter measures the "degree of nonintermittency" of the interactions. When it is equal to 1, the system is nonintermittent and K41 is recovered. The value of β in fully developed turbulence ($\beta = 2/3$) is then just the signature of the intermittency of the interactions. The present picture of intermittency holds irrespective of the details of the geometry of the dissipative structures, which are characterized by the parameter Δ [Eq. (14)]. It may then be an accident that Δ should be equal to β in isotropic fully developed turbulence. Indeed, measurements of intermittency in a turbulent boundary layer are compatible with a SL model, with a different value of Δ [6].

Several interesting issues arise from this new picture. In principle, the moment hierarchy (4) should hold in any scale-invariant system, so that a similar form of intermittency should be found in 2D turbulence or shell models of turbulence, with possibly different parameters Δ and β. Investigation of scaling properties of a class of shell models indeed confirms this possibility [12]. One may then wonder how universal Δ and β are. From the comparison between boundary layer and isotropic 3D turbulence, it is very likely that Δ is strongly influenced by the boundaries, the geometry, and external forces (as long as they do not break scale invariance). For β, the situation is less clear. The value of β found in the GOY shell model by Benzi is very close to 2/3, as in isotropic turbulence. Apart from scale invariance, the only real shared property between the shell model and 3D turbulence is energy conservation. Does it mean that β is mainly dependent on the conservation laws, and not on the dimensionality of the system? If so, we may expect different values of β to appear in 2D turbulence, characterized by different conservation laws. This point is under investigation.

I thank R. Benzi for very interesting discussions and for communicating to me his results prior to publication. I thank L. Biferale, M. Blank, Ph. Brax, U. Frisch and J. Miller for their criticisms and suggestions, E. Aurell, F-M. Bréon, and F. Graner for comments on the manuscript and J. Istas for references. This work was supported by a grant from the European Community (ERBCHRXCT920001).

Note added.—After completion of this work, I received a paper by Z.S. She and E. Waymire which discusses the log-Poisson law in relation with the general theory of indefinitely divisible processes [13].

[1] R. Benzi, S. Ciliberto, C. Baudet, R. Tripiccione, F. Massaioli, and S. Succi, Phys. Rev. E **48**, R29 (1993).

[2] A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962); A.M. Obukhov, J. Fluid Mech. **13**, 77 (1962).

[3] A.S. Gurvich and A.M. Yaglom, Phys. Fluid Suppl. "Boundary Layers and Turbulence," S59 (1967).

[4] G. Parisi and U. Frisch, in *Turbulence and Predictability*, Varenna Summer School, edited by M. Ghol, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.

[5] Z.S. She and E. Lévêque, Phys. Rev. Lett. **72**, 336 (1994).

[6] R. Benzi (private communication).

[7] E.B. Gledzer, Sov. Phys. Dokl. **18**, 216 (1973); M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn. **56**, 4210 (1987).

[8] I. Guikhman and A. Skorokhod, *Introduction to the Theory of Random Processes* (W.B. Saunders, Philadelphia, 1969).

[9] R. Arratia, L. Goldstein, and L. Gordon, Stat. Sci. **5**, No. 4, 403 (1990).

[10] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[11] U. Frisch, P-L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).

[12] P. Frick, B. Dubrulle, and A. Babiano (to be published).

[13] Z.S. She and E. Waymire, "Quantized Energy Cascades and Log-Poisson Statistics in Fully Developed Turbulence" (to be published).