Linear Stability Analysis for Bifurcations in Spatially Extended Systems with Fluctuating Control Parameter

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We study the threshold in systems which exhibit a symmetry breaking instability, described, e.g., by Ginzburg-Landau or Swift-Hohenberg equations, with the control parameter fluctuating in space and time. Because of the long-tail property of the probability distributions all the moments of the linearized equations have different thresholds and none of them coincides with the threshold of the nonlinear equation, where the long tails are suppressed. We introduce a method to obtain the threshold of the full nonlinear system from the stability exponents of the first and second moments of the linearized equation.

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The effect of noise on systems with symmetry breaking instabilities was studied experimentally during the last years in thermal convection of simple fluids [1] and binary mixtures [2], and in electrically driven instabilities in liquid crystals [3]. Corresponding theoretical investigations with additive noise involved simulations [4] and analytical approximations [5]. Moreover, the influence of spatially constant multiplicative noise in the form of temporal fluctuations of the control parameter was investigated experimentally [6,7] and theoretically [8,9]. Recently spatially distributed multiplicative noise was also treated in simulations of the two-dimensional Swift-Hohenberg equation [10] with the surprising result of a threshold shift to lower values. The analysis of the second moment (actually the structure function) in the linearized theory was found to be consistent with this result (with numerical accuracy). At present a satisfactory analytical description exists only for the zero-dimensional system [11] where a threshold shift does not occur and where the behavior of the moments (first, second, etc. or structure function) of the linearized equation is totally misleading: The moments exhibit downward shifted thresholds with larger shifts for higher moments.

In order to clarify the situation we investigate nonlinear partial differential Langevin equations for a real or complex field $\Psi(x,t)$ in a d-dimensional space ($x \in \mathbf{R}^d$):

$$\partial_t \Psi(x,t) = [D\mathcal{L}_x + a + \sqrt{\epsilon}\,\xi(x,t) - g|\Psi(x,t)|^2]\Psi(x,t), \quad (1)$$

where $\mathcal{L}_x$ is a linear operator which couples the spatial degrees of freedom. The positive parameter D measures the strength of this coupling. We are especially interested in the following cases:

$$\mathcal{L}_x = \Delta_x \ (=\text{Laplacian}), \quad \text{Ginzburg-Landua (GL)},$$

$$\mathcal{L}_x = -(1+\Delta_x)^2, \quad \text{Swift-Hohenberg (SH)}.$$

The coefficient a plays the role of a control parameter and we added the fluctuating real field $\sqrt{\epsilon}\,\xi(x,t)$ leading to a multiplicative noise. We choose Gaussian white noise which is determined by its mean and correlation function

$$\langle \xi(x,t)\rangle = 0, \quad \langle \xi(x,t)\xi(x',t')\rangle = f(x-x')\delta(t-t').$$

ϵ measures the noise strength and the function $f(x)$ gives the spatial correlation. We are especially interested in the limit $f(x) \to \delta(x)$ (small correlation length). Because of the singular nature of the white noise Eq. (1) has no differentiable solutions and the derivatives must be interpreted in the sense of stochastic calculus. In particular, for the product $\xi(x,t)\Psi(x,t)$ we use the Stratonovich interpretation relevant for physical situations. The nonlinear term $-g|\Psi(x,t)|^2\Psi(x,t)$ with $g > 0$ leads to a saturation of the instability and describes a supercritical bifurcation.

The critical behavior of the deterministic equations ($\epsilon = 0$) is well known: For negative control parameter a the only stationary solution is a trivial one, $\Psi(x,t) \equiv 0$. For positive a it becomes unstable against plane-wave solutions with wave vectors out of a band centered around the critical wave vector $\mathbf{k}_c = 0$ in the Ginzburg-Landau case and around the unit sphere $|\mathbf{k}_c| = 1$ for the Swift-Hohenberg model. This band of wave vectors widens with increasing a. These results are easily obtained from the linearized equations ($g = 0$) where one has exponential growth or decay. Questions about the structure and amplitude of the new attractors for $a > 0$ can only be answered by taking into account the nonlinearities.

The critical behavior of the stochastic equation can be studied by writing the order parameter as

$$\Psi(x,t) = u(x,t)\exp[\rho(t)] \quad (2)$$

where $u(x,t)$ is a field whose "spatial root mean modulus square" (SRMMS) is equal to 1,

$$\|u\| = \sqrt{\overline{|u(x,t)|^2}} = 1 \quad (3)$$

(the overbar denotes spatial average), so that $\exp[\rho(t)]$ is the SRMMS of $\Psi(x,t)$. It is easy to see that in the linear case ($g = 0$) $u(x,t)$ is a homogeneous Markov process on the unit hypersphere in function space, which depends

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neither on $\rho(t)$ nor on the control parameter a. $\rho(t)$ is given by

$$\rho(t) = \rho_0 + \left(a - \frac{1}{t}\int_0^t F[u(x,t'),\xi(x,t')]\,dt'\right)t. \quad (4)$$

The explicit form of the functional F can be derived easily, but is not needed in the following. The ergodic properties of the homogeneous Markov processes $u(x,t)$ and $\xi(x,t)$ show that the integral term converges for $t \to \infty$ to the stationary mean of its integrand with probability 1 [almost certain convergence (a.c.)]:

$$\text{a.c.} \lim_{t\to\infty} \frac{1}{t}\int_0^t F[u,\xi]\,dt' = \langle F[u,\xi]\rangle_{\text{st}}. \quad (5)$$

Therefore the critical value is $a_c = \langle F[u,\xi]\rangle_{\text{st}}$. For $a < a_c$ every solution $\Psi(x,t)$ converges to 0 with probability 1, whereas for $a > a_c$ every nonzero initial value leads to a divergent order parameter with probability 1. The multiplicative noise does not destroy the bifurcation (in contrast to additive noise).

The proof of the existence of this threshold was given by Khas'minskii [12]. But the calculation of a_c by the above formula is in general not possible since the stationary distribution is unknown. For a two-component system, zero-dimensional or with spatially constant noise, Müller and Behn used this method to calculate a_c [8] ("sample stability"). We introduce a method which is applicable more generally.

Since a_c is determined by the dynamics of the process $\Psi(x,t)$ for arbitrary small values of $\Psi(x,t)$, the nonlinearity has no influence on a_c. (The only qualitative change is that in the nonlinear equation the trajectories can no longer diverge for $a > a_c$.) Therefore it is possible to obtain a_c from the linearized equations alone (linear stability analysis).

The moments.—It is usually useful to study the moments $\langle\Psi(x,t)^n\rangle$ (for $n = 1,2,\ldots$). From the linearized equation (1) ($g = 0$) equations for the mean and the n-point correlation functions can be obtained by using the Ito calculus [13] or Novikov's theorem [10,14]

$$\partial_t\langle\Psi(x,t)\rangle = \left(D\mathcal{L}_x + a + \frac{\epsilon}{2}f(0)\right)\langle\Psi(x,t)\rangle, \quad (6)$$

$$\partial_t\langle\Psi(x_1,t)\Psi(x_2,t)\rangle = [D\mathcal{L}_{x_1} + D\mathcal{L}_{x_2} + 2a + \epsilon f(0) + \epsilon f(x_1 - x_2)] \times\langle\Psi(x_1,t)\Psi(x_2,t)\rangle, \quad (7)$$

$$\ldots.$$

The largest eigenvalues λ_n of the linear operators on the right hand side of these equations give the long-time behavior of the nth moment (except if the corresponding eigenfunction is zero for $x_1 = x_2 = \cdots$, then one has to take the largest eigenvalue without this property):

$$\langle\Psi(x,t)^n\rangle \sim \exp(\lambda_n t) \quad \text{for } t \to \infty. \quad (8)$$

These stability exponents characterize the critical values a_{cn} for the moments: $a \lesseqgtr a_{cn}$ for $\lambda_n \lesseqgtr 0$. (For the Ginzburg-Landau and Swift-Hohenberg equations all eigenvalues are real because the corresponding operators are self-adjoint. In general one of course has to consider the real parts of the eigenvalues.)

In the Ginzburg-Landau case ($\mathcal{L}_x = \Delta_x$) there is some similarity to quantum mechanics. There the equation for the nth moment is equivalent to an n-particle Schrödinger equation, and $na + n\epsilon f(0)/2 - \lambda_n$ is the ground-state energy of a system of n Schrödinger bosons with mass $\hbar^2/2D$, which interact pairwise via the potential $-\epsilon f(x_i - x_j)$.

We want to emphasize that in general all a_{cn} are different and none of them is identical with the previously defined threshold a_c. On the contrary one has the inequality

$$a_c \geq a_{c1} \geq a_{c2} \geq \cdots \quad (\text{for } g = 0) \quad (9)$$

and we expect the > sign to hold. This fact can be understood from the long-tail property of the probability distribution $P(\Psi; x, t)$ (the probability density for finding the value Ψ at the location x and time t): Even if the probability density is concentrated at small values of Ψ, the moments may be dominated by large values with a small probability and not by the small values around the peak of the probability distribution. This can lead to diverging moments although the probability distribution converges to a δ peak at zero. Since this effect becomes stronger for larger exponents n, the moments of higher order diverge earlier. This explains the inequality for the critical values a_c, a_{cn}.

The situation changes abruptly when the nonlinearity is taken into account because then the long tails of the probability distributions are suppressed. [So the averages for the moments are no longer dominated by large Ψ values and the moments will converge to zero as long as the probability density approaches $\delta(\Psi)$.] Therefore we expect

$$a_c = a_{c1} = a_{c2} = \cdots \quad (\text{for } g \neq 0). \quad (10)$$

Consequently, the thresholds for the moments of the linearized equation in general only give lower bounds to the threshold for the nonlinear system. Only the critical value a_c for the probability density as defined before does not depend on the nonlinearity.

These facts have been proven directly for the zero-dimensional equation

$$\dot{\Psi}(t) = [a + \sqrt{\epsilon}\,\xi(t)]\Psi(t) - g\Psi(t)^3,$$
$$\langle\xi(t)\rangle = 0, \quad \langle\xi(t)\xi(t')\rangle = \delta(t - t'). \quad (11)$$

From the linearized equation one obtains for the moments

$$\frac{d}{dt}\langle\Psi(t)^n\rangle = \left(na + \frac{n^2}{2}\epsilon\right)\langle\Psi(t)^n\rangle \quad (12)$$

and therefore the critical values are

$$a_{cn} = -\frac{n}{2}\epsilon, \qquad (13)$$

whereas the exact solution of the linearized equation is

$$\Psi(t) = \Psi_0 \exp\left(at + \sqrt{\epsilon}\int_0^t \xi(s)\,ds\right)$$
$$= \Psi_0 \exp\left[\left(a + \sqrt{\epsilon}\frac{W(t)}{t}\right)t\right]. \qquad (14)$$

Since the Wiener process $W(t) = \int_0^t \xi(s)\,ds$ fulfills $W(t)/t \to 0$ for $t \to \infty$ with probability 1 we get (for $t \to \infty$ with probability 1)

$$\Psi(t) \to 0, \quad \text{if } a < 0,$$
$$\Psi(t) \text{ diverges}, \quad \text{if } a > 0,$$

and therefore $a_c = 0$. Also the nonlinear equation can be solved rigorously [11] with the result

$$a_c = a_{cn} = 0 \quad \text{for all } n \in \mathbf{N}. \qquad (15)$$

Calculation of a_c (linear stability analysis).—Since the full information about the stochastic process is contained in its moments, we could calculate the threshold a_c if we had the long-time solutions for all moments. But this would be a formidable task. Instead we look for a simple approximation for the long-time limit which describes the long-tail behavior of the probability distribution correctly and whose parameters can be determined from the first few moments. Our choice is motivated by the zero-dimensional case, Eq. (14), where $\Psi(t)$ is given by the exponential of a Gaussian distributed process with mean at and variance ϵt. Since the corresponding probability density gives the correct long-tail behavior, we make for the systems with spatial degrees of freedom the similar ansatz that $\rho(t)$ is a Gaussian process, whose mean and variance will be called $m(t)$ and $\sigma(t)$. It can be shown that this assumption leads to correct results for the critical value a_c up to the second order in the noise strength (ϵ^2) (under some weak conditions which are fulfilled in the systems mentioned) [15]. Then the first and second moments of the SRMMS of the order parameter $\Psi(x,t)$ are

$$\langle\|\Psi(x,t)\|\rangle = \exp\left[m(t) + \tfrac{1}{2}\sigma(t)\right],$$
$$\langle\|\Psi(x,t)\|^2\rangle = \exp[2m(t) + 2\sigma(t)]. \qquad (16)$$

Expressing $m(t)$ and $\sigma(t)$ in terms of these two moments leads to

$$m(t) = 2\ln\langle\|\Psi(x,t)\|\rangle - \tfrac{1}{2}\ln\langle\|\Psi(x,t)\|^2\rangle,$$
$$\sigma(t) = \ln\langle\|\Psi(x,t)\|^2\rangle - 2\ln\langle\|\Psi(x,t)\|\rangle. \qquad (17)$$

Since the SRMMS of the order parameter has the same long-time characteristic as the order parameter itself, we can use the long-time behavior of the moments from Eq. (8). We obtain the result (for $t \to \infty$)

$$m(t) = \left(2\lambda_1 - \tfrac{1}{2}\lambda_2\right)t,$$
$$\sigma(t) = (\lambda_2 - 2\lambda_1)t. \qquad (18)$$

Clearly the threshold is given by $m = 0$, i.e.,

$$a = a_c \Longleftrightarrow 2\lambda_1 = \tfrac{1}{2}\lambda_2. \qquad (19)$$

Results.—We performed the explicit calculations for the Ginzburg-Landau and Swift-Hohenberg equations in one and two dimensions. For the first moment one obtains in all four cases

$$\lambda_1 = a + \frac{\epsilon}{2}f(0), \quad a_{c1} = -\frac{\epsilon}{2}f(0), \qquad (20)$$

as can be seen immediately from Eq. (6). The calculation of the stability exponent for the second moment from Eq. (7) leads to nontrivial eigenvalue problems which we do not discuss in detail here. For $f(x) \to \delta(x)$ (or, more generally, for correlation length small compared to D/ϵ) the results are

$$\text{GL 1D}: \quad \lambda_2 = 2\left(a + \frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{16D}\right),$$
$$\text{2D}: \quad \lambda_2 \approx 2\left[a + \frac{\epsilon}{2}f(0) + \pi^2 D f(0) \times \exp\left(-\frac{8\pi D}{\epsilon}\right)\right],$$
$$\text{SH 1D}: \quad \lambda_2 = 2\left(a + \frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{16D} + O(\epsilon^3)\right),$$
$$\text{2D}: \quad \lambda_2 = 2\left(a + \frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{64D} + O(\epsilon^3)\right). \qquad (21)$$

Note that the term $f(0)$ becomes singular in the limit $f(x) \to \delta(x)$. For the two-dimensional Ginzburg-Landau equation the spectrum of the eigenvalue problem (7) has no upper bound for $f(x_1 - x_2) = \delta(x_1 - x_2)$; we got our result by a formal cutoff in Fourier space with $k_{\max}^2 = \pi^2 f(0)$ which is motivated by considering the problem on a lattice with discretization Δx. Note that the threshold shifts for the first and second moments are of order ϵ (as in the zero-dimensional case), but their difference is of higher order (in contrast to the zero-dimensional case), see Eq. (13). With Eq. (19) we finally get for the critical values of the control parameter

$$\text{GL 1D}: \quad a_c \approx -\frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{16D},$$
$$\text{2D}: \quad a_c \approx -\frac{\epsilon}{2}f(0) + \pi^2 D f(0)\exp\left(-\frac{8\pi D}{\epsilon}\right),$$
$$\text{SH 1D}: \quad a_c \approx -\frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{16D},$$
$$\text{2D}: \quad a_c \approx -\frac{\epsilon}{2}f(0) + \frac{\epsilon^2}{64D}, \qquad (22)$$

which again differ from a_{c1} only by terms of order ϵ^2 or higher. For the two-dimensional Ginzburg-Landau equation this shift is actually proportional to $f(0)$ but of infinitely small order in ϵ since $\exp(-8\pi D/\epsilon)$ is not analytical in $\epsilon = 0$.

In order to check our results we did Monte Carlo simulations for the one-dimensional Ginzburg-Landau equation. The Laplace operator was calculated in Fourier space (pseudospectral code). Because of the multiplicative noise we could only use an algorithm of first order in Δt, so we had to choose a rather small time step: $\Delta t = 0.001$. For $\epsilon = 1$ and discretization $\Delta x = 0.1$, i.e., $f(0) = \delta(0) = (\Delta x)^{-1} = 10$, our theory predicts $a_c = -4.9375$ whereas $a_{c1} = -5$. The simulation gave $-4.940 < a_c < -4.935$ for both the linear equation $g = 0$ and the nonlinear $g = 1$ in perfect agreement with our theory.

The rather large leading-order threshold shift is consistent with the numerical results of [10], but there the accuracy was not sufficient to verify the contribution of order ϵ^2. For strong spatial coupling ($D \to \infty$) this contribution vanishes, and the critical value a_c becomes equal to that of the first moment of the linearized equation a_{c1}. Neither the zero-dimensional system Eq. (11) nor Eq. (1) with spatially constant noise (which is in the linearized form actually equivalent to the zero-dimensional problem, as can be seen by Fourier transformation) exhibit such a shift. In the limit of weak spatial coupling ($D \to 0$) the system becomes equivalent to the zero-dimensional one, i.e., the shift vanishes. The expansion of λ_2 and a_c in orders of ϵ breaks down then, as can be seen from the diverging contributions of order ϵ^2 in (21) and (22). In fact the shift results from an interaction between the spatial coupling and the spatially nonconstant multiplicative noise. Actually no continuum is necessary to obtain this effect and already two coupled ordinary differential equations with uncorrelated multiplicative noise show the shift. Thus for the system

$$\dot{u} = [a + \sqrt{\epsilon}\,\xi_1(t)]u + v - u,$$
$$\dot{v} = [a + \sqrt{\epsilon}\,\xi_2(t)]v + u - v,$$
$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t - t'), \qquad (23)$$

our method gives

$$a_c = -\frac{\epsilon}{4} + \frac{\epsilon^2}{32} + O(\epsilon^4). \qquad (24)$$

This problem can be treated rigorously with the method from Ref. [12] which gives the same result up to order ϵ^2, as expected.

Clearly our method for calculating the threshold a_c from the stability exponents of the first and second moments of the linearized equation is not restricted to the equations mentioned in this Letter. We expect that it works similarly for many other models, even for equations with higher-order time derivatives or multicomponent systems.

An analysis for spatially correlated noise, which should give the connection between spatially constant and δ-correlated noise, will be published elsewhere. Finally we note that understanding the nonlinear behavior above threshold presents a very challenging problem. In the zero-dimensional case (Stratonovich model) the problem was solved rigorously for Gaussian white noise [11] and dichotomous noise [16].

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