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Reversible Quantum Measurements on a Spin 1/2 and Measuring the State of a Single System

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A measurement procedure on a spin 1/2 system is described, whose effect on the (possibly unknown) state of the system can be reversed, by means of other similar measurements, with a sizable probability of success P^{rev} (e.g., $1/2 < P^{\text{rev}} < 1$, depending on the state). Repeated such measurements followed by reversal, on a *single* system, allow us to determine its (*a priori* unknown) state operator with a probability of success which is nonzero, though extremely small (because $P^{\text{rev}} < 1$).

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(1) Introduction.—Irreversibility is a notorious aspect of quantum measurements: In particular, every quantum measurement must necessarily terminate with the (thermodynamically) irreversible amplification of some microscopic quantum event into a macroscopically observable fact (e.g., the visible mark made by a single photon on a photographic plate). von Neumann [1] pointed out another, more specifically quantum, form of irreversibility: Consider measurements of the first kind, of an observable $\hat{A} = \sum_n |\phi_n\rangle a_n \langle\phi_n|$, on an ensemble of systems all in the same (initial) state $|\Psi^i\rangle$, hence with a *pure* density operator $\hat{\rho}^i = |\Psi^i\rangle\langle\Psi^i|$. Since the measurements "collapse" each individual system into one of the states $|\phi_n\rangle$ with probability $|\langle\phi_n|\Psi^i\rangle|^2$, one ends up with an ensemble described by a *mixed* (final) density operator $\hat{\rho}^f$:

$$\hat{\rho}^i = |\Psi^i\rangle\langle\Psi^i| \longrightarrow \hat{\rho}^f = \sum_n |\phi_n\rangle\langle\phi_n|\hat{\rho}^i|\phi_n\rangle\langle\phi_n|. \quad (1.1)$$

Unless $|\Psi^i\rangle$ belongs to the orthogonal set $\{|\phi_n\rangle\}$, in which case $\hat{\rho}^f = \hat{\rho}^i$, the process (1.1) is irreversible in *this* sense that no *unitary* transformation (which necessarily preserves the eigenvalues of $\hat{\rho}$) can convert $\hat{\rho}^f$ back into $\hat{\rho}^i$. Also, the "entropy" of the ensemble, $S = -k_B \mathrm{Tr}\{\hat{\rho}\ln\hat{\rho}\} = -k_B \sum_j \rho_j \log \rho_j$ (ρ_j the eigenvalues of $\hat{\rho}$) increases under a transformation (1.1) from a pure state (one of the $\rho_j = 1$, all others zero, $S = 0$) to a mixed state ($S > 0$).

The above irreversibility is really a bit artificial: Recall that $\hat{\rho}^f$ in (1.1) describes an ensemble of systems, each in a pure state $|\phi_n\rangle$ *which is known* (from the readouts of the individual measurements). So in fact, the measurements can be reversed *system by system,* since any two pure states, such as $|\phi_n\rangle$ and $|\Psi^i\rangle$, can always be connected by a unitary transformation. Suppose, for instance, we measure the z spin component, $\hat{S}_z$, on an ensemble of spins 1/2 all in state $|+,x\rangle$ (we denote by $|\pm,x\rangle$ the eigenkets of $\hat{S}_x$, etc.): Then each spin gets collapsed into either $|+,z\rangle$ or $|-,z\rangle$, which can indeed be unitarily transformed back into the initial state $|+,x\rangle$, by either a $-\pi/2$ or a $+\pi/2$ rotation about the y axis. Note that different final states require different rotations. So indeed, as noted by von Neumann, no *single* unitary transformation can convert *all* the individual final (postmeasurement) states, or the density matrix which represents them as an ensemble, back into the initial (premeasurement) state. But each individual measurement *is* reversible, *provided the initial states are known* [2].

If, however, an initial state is *unknown,* then we cannot revert back to it (knowingly), because the unitary transformation required depends on it; and if we do so by luck (e.g., we rotate a final state $|+,z\rangle$ into $|+,x\rangle$, say, and the initial state happened to be $|+,x\rangle$, but unknown to us), then we cannot be aware of it. Moreover, there can be no hope of ever (knowingly) getting back an unknown initial state, because all information on it has been lost: Indeed, in a measurement of the first kind, as used here, the final state is known, but it has no "memory," that is, it carries no information on the initial state (apart that the latter had

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nonzero overlap with it). Thus, *a measurement of the first kind on a* single *system is irreversible in the sense that one cannot knowingly get back the initial state if the latter was not known a priori.* This is the sort of (ir)reversibility which we have in mind in this Letter.

Recently, Ueda and Kitagawa [3] introduced the notion of "logically reversible" measurements, in which the final state retains complete information on the initial state, and gave a specific example. Imamoglu [4] subsequently described a quantum nondemolition (QND), logically reversible measurement; he also raised the possibility of *physically* reversible measurements, which might allow one to measure the quantum state of a single system. This would be quite different from the "state measurement" described by Aharonov and co-workers [5], by means of "weak" measurements on a "protected" state, which requires that the state essentially be known *a priori.*

In this Letter, we describe a measurement on a spin 1/2 system, having the following properties: (i) It transforms the initial state in a known, information preserving manner (it is logically reversible), but the final state is not known, if the initial state is not. (ii) Its effect on the system can be reversed (knowingly) by subsequent measurements of the same kind (even if the initial state is unknown), but only with a probability of success $P^{\rm rev}$ less than 1, though sizable (e.g., $1/2 < P^{\rm rev} < 1$). (iii) Measurements on many systems in the same state $\hat{\rho}^i$ allow one to deduce the diagonal elements $\langle \pm|\hat{\rho}^i|\pm\rangle$. (iv) Repeated measurements followed by reversal on a *single* system allow one to determine its state operator $\hat{\rho}^i$ with a probability of success $(P^{\rm rev})^N$, where N is the number of measurements (hence reversals) required to achieve some prescribed accuracy; that probability is very small (because $P^{\rm rev} < 1$), but finite.

(2) States of a spin 1/2.—Let us first recall some basic features of spin 1/2 systems, and establish some notation: Let $\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ be the spin operator for a spin 1/2. We shall usually denote the eigenstates $|\pm, z\rangle$ of $\hat{S}_z$ simply by $|\pm\rangle$. The most general normalized *pure* state is of the form

$$|\Psi\rangle = |+\rangle c_+ + |-\rangle c_-, \quad |c_+|^2 + |c_-|^2 = 1\,. \tag{2.1}$$

Pure states, however, are not the natural states for systems in the wild: More usually, a system s will be in a mixed state, due to entanglement with the environment e (in full generality, the rest of the Universe) with which it must have interacted in the past. To obtain a pure state, in fact, one has to break any entanglement with e by an act of observation, collapsing s into some (known) such state. In this Letter, however, we are concerned with measurements on *unknown,* hence likely entangled, states. The most general normalized entangled state for a spin 1/2 system s is of the form

$$\begin{gathered} |\Psi\rangle_{s+e} = |+\rangle_s|\chi_+\rangle_e + |-\rangle_s|\chi_-\rangle_e\,, \\ \langle\chi_+|\chi_+\rangle + \langle\chi_-|\chi_-\rangle = 1\,. \end{gathered} \tag{2.2a}$$

The "reduced" state operator of s alone (obtained by tracing out the environment) is

$$\begin{aligned} \hat{\rho} &= \mathrm{Tr}_e\{|\Psi\rangle_{s+e}\langle\Psi|\} \\ &= \begin{pmatrix} \langle\chi_+|\chi_+\rangle & \langle\chi_-|\chi_+\rangle \\ \langle\chi_+|\chi_-\rangle & \langle\chi_-|\chi_-\rangle \end{pmatrix}_z \equiv \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}_z, \end{aligned} \tag{2.2b}$$

where a subscript z on a matrix indicates that it is a representation in the $|\pm, z\rangle$ basis. The representations of (2.2) in the bases $|\pm, x\rangle$ and $|\pm, y\rangle$ are

$$\begin{aligned} \hat{\rho} &= \begin{pmatrix} \frac{1}{2} + \mathrm{Re}\{c\} & \left(\frac{1}{2} - a\right) + i\,\mathrm{Im}\{c\} \\ \left(\frac{1}{2} - a\right) - i\,\mathrm{Im}\{c\} & \frac{1}{2} - \mathrm{Re}\{c\} \end{pmatrix}_x \\ &= \begin{pmatrix} \frac{1}{2} - \mathrm{Im}\{c\} & \left(\frac{1}{2} - a\right) + i\,\mathrm{Re}\{c\} \\ \left(\frac{1}{2} - a\right) - i\,\mathrm{Re}\{c\} & \frac{1}{2} + \mathrm{Im}\{c\} \end{pmatrix}_y. \end{aligned} \tag{2.3}$$

The probabilities that a system be "found" in $|+, z\rangle$ or $|-, z\rangle$ are equal to a and b, so that the latter can be deduced by measuring $\hat{S}_z$ on many individual systems of a statistical ensemble. Also, measurements of $\hat{S}_x$ and $\hat{S}_y$ yield the quantities $\frac{1}{2} \pm \mathrm{Re}\{c\}$ and $\frac{1}{2} \mp \mathrm{Im}\{c\}$, in view of (2.3): Thus, measurements of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ together allow us to determine $\hat{\rho}$.

(3) Quantum measurements.—Before describing our reversible measurements, let us recall, for context, the underlying process in measurements of the first kind, and motivate the overall approach we will use.

Any knowledge about a microsystem s must ultimately be inferred from a *macroscopic* change, or absence of such a change, in a (macroscopic) apparatus M. For s to *directly* trigger a change in the *macrostate* of M, it must interact "strongly" with M, and thereby "feel" its *intrinsically unpredictable microstate.* Therefore, the state of s will itself be largely unpredictable after such an action. So, if one wishes to "control" s, then it must not interact strongly with M. Thus, to *prepare* s in a definite state, one often resorts to a "negative measurement," whereby collapse into the desired state is produced by the *macroscopically observable absence* of an interaction: For instance, to prepare a spin 1/2 system s in a $|+\rangle$ state, pass it through a Stern-Gerlach experiment with the $|-\rangle$ channel blocked by a screen; if no mark appears on the screen, then s has collapsed into $|+\rangle$ (if, on the contrary, a mark appears, then s has been "found" in the state $|-\rangle$, i.e., the measurement readout is $|-\rangle$, but the precise state of s thereafter is unpredictable).

The above negative measurement is suitable for collapsing s into *one* definite state. But it is inadequate if one wants the final state to be predictable for *all* possible measurement readouts ($|+\rangle$ or $|-\rangle$), as in a measurement of the first kind. To achieve this feat, it seems the only possible way is to let s interact strongly only with another microscopic "probe" (or "messenger") object m whose state is controllable (e.g., a photon), with which it gets entangled; then, *after* the s-m interaction is off, m will interact strongly with an apparatus (e.g., a photographic plate), yielding a measurement readout, and collapsing s into a definite state. The prototypical probe measurement was formulated by von Neumann [1], in a model of position measurement with a "meter" (which we prefer to call a probe, because meter rather evokes a macroscopic object).

(4) A measurement scheme.—We will now devise a measurement scheme on a spin 1/2 system, in which the final state can preserve complete information on the initial state. Let $\hat{S}_s = (\hat{S}_{sx}, \hat{S}_{sy}, \hat{S}_{sz})$ be the spin operator for the spin 1/2 system s on which we intend to perform measurements. We suppose s is initially in an (unknown) entangled state $|\Psi^i\rangle_{s+e}$ of the general form (2.2). Since we want to "control" the state of s, we let it interact only with a microscopic probe, whose state is controllable. We use as probe another spin 1/2 system m, with spin operator $\hat{S}_m = (\hat{S}_{mx}, \hat{S}_{my}, \hat{S}_{mz})$, prepared in an initial pure state (e.g., by means of a negative measurement)

$$|\theta\rangle_m = e^{-i\theta\hat{S}_{my}}|+\rangle_m = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle_m + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle_m \tag{4.1}$$

that is, a $|+,z\rangle$ state rotated by an angle θ about the y axis. The initial state of the combined probe-system environment is thus $|\theta\rangle_m|\Psi^i\rangle_{s+e}$. We let s interact impulsively with m via a Hamiltonian $g(t)\hat{S}_{sz}\hat{S}_{my}$, $\int dt\, g(t) = \sigma$, which dominates the time evolution during the short interval where $g(t) \neq 0$, so that the unitary evolution operator connecting states before and after the interaction is

$$\hat{U} = e^{-i\sigma\hat{S}_{sz}\hat{S}_{my}}. \tag{4.2}$$

The interaction of m with s entangles it with $s + e$: Indeed, the state of $m + s + e$ after the interaction is

$$\hat{U}|\theta\rangle_m|\Psi^i\rangle_{s+e} = e^{-i(\theta+\sigma\hat{S}_{sz})\hat{S}_{my}}|+\rangle_m|\Psi^i\rangle_{s+e}, \tag{4.3a}$$

$$= \{\hat{T}_1|+\rangle_m + \hat{T}_2|-\rangle_m\}|\Psi^i\rangle_{s+e}, \tag{4.3b}$$

$$= |+\rangle_m|\tilde{\Psi}_{T_1}\rangle_{s+e} + |-\rangle_m|\tilde{\Psi}_{T_2}\rangle_{s+e}, \tag{4.3c}$$

where

$$\hat{T}_1 = \cos\left(\tfrac{1}{2}\theta + \tfrac{1}{2}\sigma\hat{S}_{sz}\right), \quad \hat{T}_2 = \sin\left(\tfrac{1}{2}\theta + \tfrac{1}{2}\sigma\hat{S}_{sz}\right) \tag{4.4}$$

are nonunitary Hermitian operators acting on s alone, and $|\tilde{\Psi}_T\rangle_{s+e}$ (we let T, without a subscript, stand for either T_1 or T_2) are *unnormalized* (indicated by a twidle) entangled states given by

$$|\tilde{\Psi}_T\rangle_{s+e} = \hat{T}|\Psi^i\rangle_{s+e} = T_+|\chi_+\rangle_e|+\rangle_s + T_-|\chi_-\rangle_e|-\rangle_s. \tag{4.5}$$

Here, we wrote (since $\hat{S}_z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle$)

$$\hat{T} = |+\rangle T_+\langle+| + |_-\rangle T_-\langle-| = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}_z, \tag{4.6}$$

where

$$T_{1\pm} = \cos\left(\tfrac{1}{2}\theta \pm \tfrac{1}{4}\sigma\right), \quad T_{2\pm} = \sin\left(\tfrac{1}{2}\theta \pm \tfrac{1}{4}\sigma\right), \tag{4.7a}$$

$$T_{1+}^2 + T_{2+}^2 = T_{1-}^2 + T_{2-}^2 = 1. \tag{4.7b}$$

We now measure $\hat{S}_{mz}$, the z spin of the probe m. This may be done by passing m through a Stern-Gerlach experiment and recording its arrival on a screen (its state thereafter is unpredictable, but also irrelevant for us). According as m is "found" in $|+\rangle$ or $|-\rangle$ (i.e., makes a mark on the top or bottom of the screen), $s + e$ gets "collapsed" into $|\tilde{\Psi}_{T_1}\rangle_{s+e}$ or $|\tilde{\Psi}_{T_2}\rangle_{s+e}$ [in view of (4.3c)], with respective probabilities P_{T_1} and P_{T_2} given by

$$P_T = \langle\tilde{\Psi}_T|\tilde{\Psi}_T\rangle = T_+^2 a + T_-^2 b = (T_+^2 - T_-^2)a + T_-^2, \tag{4.8}$$

where we used the notation (2.2c), namely $\langle\chi_+|\chi_+\rangle = a$, $\langle\chi_-|\chi_-\rangle = b$, and $a + b = 1$.

Equations (4.5) and (4.8) may be expressed in terms of state operators for s alone [see (2.2b)], namely, $\tilde{\rho}_T = \mathrm{Tr}_e\{|\tilde{\Psi}_T\rangle\langle\tilde{\Psi}_T|\} = \hat{T}\mathrm{Tr}_e\{|\Psi^i\rangle\langle\Psi^i|\}\hat{T} = \hat{T}\hat{\rho}^i\hat{T}$ (recall that $\hat{T}^\dagger = \hat{T}$), or, after normalization to unit trace,

$$\hat{\rho}_T = (P_T)^{-1}\hat{T}\hat{\rho}^i\hat{T} = \frac{1}{T_+^2 a + T_-^2 b}\begin{pmatrix} T_+^2 a & T_+T_-c \\ T_+T_-c^* & T_-^2 b \end{pmatrix}_z \equiv \begin{pmatrix} a_T & c_T \\ c_T^* & b_T \end{pmatrix}_z, \tag{4.9}$$

$$P_T = \mathrm{Tr}\{\hat{T}\hat{\rho}^i\hat{T}\} = T_+^2 a + T_-^2 b. \tag{4.10}$$

We could, in fact, have carried the argument solely in terms of the state operator of s, without referring explicitly to the environment e. But it was of interest to see how the entanglement with e is carried along, and gets *modulated* [Eq. (4.5)] by the measurement, without being broken (if $T_\pm \neq 0$), unlike in a measurement of the first kind [see Eq. (4.11) below].

The hallmark of the above procedure is the pair of transformations $\{T_1, T_2\}$, which completely determine the possible final states (4.9), and their probabilities (4.10), in terms of the initial state $\hat{\rho}^i$. The procedure will therefore be called a $\{T_1, T_2\}$ measurement. A specific outcome will be identified by its resulting transformation, T_1 or T_2, which is known from the measurement readout. Note that the final state itself (with which we would identify the outcome of a measurement of the first kind) is not known here (if $T_{1,2\pm} \neq 0$), unless the initial state is.

The real parameters $T_{1\pm}$, $T_{2\pm}$ can be given any desired values, subject to (4.7b) (which guarantee that $P_{T_1} + P_{T_2} = 1$), by suitable choice of the angle θ and interaction strength σ in (4.1) and (4.2). For instance, to produce a measurement of the first kind, as described in Sec. 3, put $\theta = \frac{1}{2}\pi$ and $\sigma = -\pi$: Then, by (4.6) and (4.7),

$$\hat{T}_1 = |+\rangle\langle+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_z, \quad \hat{T}_2 = |-\rangle\langle-| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_z, \tag{4.11}$$

are just projectors, so that indeed $\hat{\rho}_{T_1} = |+\rangle\langle+|$, $P_{T_1} = a$, and $\hat{\rho}_{T_2} = |-\rangle\langle-|$, $P_{T_2} = b$. Here, the final states are known, but contain no information on the initial state.

(5) Reversible measurements.—Let us henceforth assume that [unlike in the first kind case (4.11)]

$$T_{1\pm} \neq 0, \quad T_{2\pm} \neq 0, \tag{5.1a}$$

$$T_{1+}^2 \neq T_{1-}^2 \Longleftrightarrow T_{2+}^2 \neq T_{2-}^2 . \tag{5.1b}$$

Equations (5.1a) imply that $\hat{T}$ in (4.9) is invertible, so that the initial state can be deduced from the final state:

$$\hat{\rho}^i = \frac{\hat{T}^{-1}\hat{\rho}_T\hat{T}^{-1}}{\mathrm{Tr}\{\hat{T}^{-1}\hat{\rho}_T\hat{T}^{-1}\}} . \tag{5.2}$$

Thus, the measurement is logically reversible in the sense of Ref. [3]. Of course, the final state is not known (unless the initial state is): Still, the procedure is justifiably called a measurement, for it yields information on $\hat{\rho}^i$: Indeed, (5.1b) imply that Eqs. (4.8) can be solved for a and b in terms of the (measurable) probabilities P_{T_1} and P_{T_2}.

Since the final state (4.9) after a $\{T_1, T_2\}$ measurement contains complete information on the initial state, one may wonder whether it is possible to knowingly get back that (unknown) initial state: This cannot be done by a unitary transformation, which would have to depend on $\hat{\rho}^i$ and $\hat{\rho}_T$, which are not known. Also, although $\hat{T}^{-1}$ exists, it is not of the physically realizable form (4.6), since $|T_{\pm}^{-1}| > 1$ [by (4.7) and (5.1a)]. However, to reverse the measurement, one does not need $\hat{T}^{\mathrm{rev}}\hat{T} = \hat{1}$, but only

$$\hat{T}^{\mathrm{rev}}\hat{T} = \alpha\hat{U}, \quad \hat{T} = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}_z , \tag{5.3}$$

where α is any complex number, and $\hat{U}$ any unitary operator, since $|\Psi\rangle$ and $\alpha|\Psi\rangle$ represent the same state, and $\hat{U}$ can be undone by its inverse (namely a rotation). The only operators of the form (4.6) which satisfy (5.3) are [6]

$$\hat{T}^{\mathrm{rev}} = k\begin{pmatrix} T_- & 0 \\ 0 & \pm T_+ \end{pmatrix}_z, \quad \hat{T}^{\mathrm{rev}}\hat{T} = kT_+T_-\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}_z, \tag{5.4}$$

where k is any real number such that $|kT_{\pm}| \leq 1$. It follows that by means of a second measurement $\{T^{\mathrm{rev}}, T''\}$, the first measurement can be reversed with a probability of success P^{rev} equal to the probability for the outcome T^{rev}, namely,

$$P^{\mathrm{rev}} = k^2T_-^2a_T + k^2T_+^2b_T = \frac{k^2T_+^2T_-^2}{T_+^2a + T_-^2(1-a)} \tag{5.5}$$

[we used (4.9), (4.10) and $a + b = 1$]. Since $0 \leq a \leq 1$, $P^{\mathrm{rev}} = P^{\mathrm{rev}}(a)$ lies between $k^2T_+^2$ and $k^2T_-^2$, depending on the (unknown) value of a. For example, if $T_+^2 = 1/2$, $T_-^2 = 2/3$, and $k^2 = 1$, then $1/2 \leq P^{\mathrm{rev}} \leq 2/3$. With $k^2 = 3/2$, we get $3/4 \leq P^{\mathrm{rev}} \leq 1$; in that case, however, $(T_+^{\mathrm{rev}})^2 = 1$, implying $(T_+'')^2 = 0$ [by (4.7b)], so that if the outcome is T'', then s collapses into $|-\rangle$, losing all memory of $\hat{\rho}^i$. To avoid this, we must take k such that $|kT_{\pm}| < 1$ (strictly).

If we do not achieve reversal after the second measurement (the outcome was $\hat{T}''$ rather than $\hat{T}^{\mathrm{rev}}$), we can still try to get back ρ^i by doing additional, suitably tuned $\{T_1, T_2\}$ measurements, thereby increasing the probability of reversal P^{rev} above the one shot value (5.5). But one can show that P^{rev} always remains less than 1, whatever the number of measurements made.

An interesting special case is if

$$\hat{T}_1 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_z, \quad \hat{T}_2 = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}_z \quad (0 \neq A \neq B \neq 0). \tag{5.6}$$

For instance, by putting $\theta = \frac{1}{2}\pi$, $0 < |\sigma| < \pi$ in (4.7), we get (5.6) with $A = \cos(\frac{1}{4}\pi + \frac{1}{4}\sigma)$, $B = \cos(\frac{1}{4}\pi - \frac{1}{4}\sigma)$ [note that $|\sigma| = \pi$ would yield the nonreversible first kind case (4.11)]. We then have $\hat{T}_2\hat{T}_1 = \hat{T}_1\hat{T}_2 = AB\hat{1}$, that is, the measurement is *self-reversing:* One gets back the initial state whenever, in a sequence of measurements, the outcomes T_1 and T_2 occur in equal numbers.

(6) Measuring a state operator.—By doing $\{T_1, T_2\}$ measurements on individual systems of a statistical ensemble, one can determine its density operator $\hat{\rho}^i$, similarly as was done in Sec. (2) by means of measurements of the first kind: One first deduces a (and $b = 1 - a$) from the relative frequency P_{T_1} of outcomes T_1, using (4.8). By then doing measurements with $\hat{S}_{sz}$ in (4.2)–(4.4) replaced by $\hat{S}_{sx}$, and then by $\hat{S}_{sy}$, we determine Re$\{c\}$ and Im$\{c\}$ [in view of (2.3)], whence $\hat{\rho}^i$.

Suppose that, in the above, N measurements are required to achieve some prescribed accuracy. It follows that by repeated measurements, followed by reversal, on a *single* system, one can measure its state operator with a probability of success $(P^{\mathrm{rev}})^N$. Although finite, this is very small, because $P^{rev} < 1$. Still, it appears from the above that one can no longer affirm that it is *in principle* impossible to measure the quantum state of a single system [7].

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ., Princeton, NJ, 1955).

[2] This remains true even if the initial states are mixed due to entanglement with the environment.

[3] M. Ueda and M. Kitagawa, Phys. Rev. Lett. **68**, 3424 (1992).

[4] A. Imamoglu, Phys. Rev. A **47**, R4577 (1993).

[5] Y. Aharonov and L. Vaidman, Phys. Lett. A **178**, 38 (1993); Y. Aharonov, J. Anandan, and L. Vaidman, Phys. Rev. A **47**, 4646 (1993).

[6] This is true even if we allow $\hat{T}^{rev}$ to be of the form $\hat{T}^{rev} = \hat{R}\hat{T}'\hat{R}^\dagger$ where $\hat{T}'$ is diagonal in $|\pm, z\rangle$, and $\hat{R}$ is any (unitary) rotation operator. Indeed, $\hat{R}\hat{T}'\hat{R}^\dagger\hat{T} = \alpha\hat{U}$ implies $\hat{T}'\hat{R}^\dagger\hat{T} = \alpha\hat{U}'$ where $\hat{U}' = \hat{R}^\dagger\hat{U}$ is unitary, so that $(\hat{T}'\hat{R}^\dagger\hat{T})(\hat{T}\hat{R}\hat{T}') = |\alpha|^2\hat{1}$, whence $\hat{R}^\dagger\hat{T}^2\hat{R} = |\alpha|^2(\hat{T}')^{-2}$. Since $\hat{T}^2$ and $(\hat{T}')^{-2}$ are both diagonal in z, $\hat{R}$ must leave the z axis fixed. Thus, $\hat{T}^{rev}$ must be diagonal in z. Hence, so is $\hat{T}^{rev}\hat{T} = \alpha\hat{U}$. But the only unitary matrices diagonal in z are $\pm\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, whence (5.4).

[7] See, e.g., A. Royer, Phys. Rev. Lett. **55**, 2745 (1985); Found. Phys. **19**, 3 (1989).