Quantum Hall Ferromagnets

Tin-Lun Ho
*Physics Department, The Ohio State University, Columbus, Ohio 43210**
and National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32306-4005
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It was pointed out recently that the $\nu = 1/m$ quantum Hall states in bilayer systems behave like easy plane quantum ferromagnets. We study the magnetotransport of these systems using their "ferromagnetic" properties and a novel spin-charge relation of their excitations. The general transport is a combination of the usual Hall transport and a time dependent transport with *quantized* time average. The latter is due to a phase slippage process in *spacetime* and is characterized by two topological constants.

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Recent technological advances in producing high mobility bilayers have provided opportunities for studying quantum Hall (QH) systems with internal degrees of freedom [1]. In this case, the internal degrees of freedom is a pseudo-spin 1/2 labeling the upper and lower layers. There is experimental evidence [2] indicating that under appropriate conditions, QH states (at $\nu = 1/2$ and $\nu = 1$) can be stabilized by correlations between layers and have little to do with tunneling between them. It is conceivable that similar correlated states will be found at lower filling factors. Bilayer QH states with $\nu = 1/m$ (m odd) are believed to be the so-called (mmm) state. It was realized recently that they are very unusual QH fluids. Wen and Zee [3] showed that these states can produce Josephson-like tunneling between the layers. Very recently, a number of authors [4] have pointed out that these states behave like easy plane quantum ferromagnets and have studied possible phase transitions in these systems.

The purpose of this paper is to study the magnetotransport of QH ferromagnets (QHFs). The difficulty of the study arises from the fact that the usual procedure of extracting conductivity tensors (say, using flux insertion arguments or Chern-Simon type theories) breaks down when applied to the (mmm) state [5]. Here, we take a new approach which amounts to studying the hydrodynamics of QHFs. Our approach recovers the known results for ordinary QH fluids, and reveals a new transport mode in QHF. A general QHF transport is a combination of usual QH transport and a genuinely time dependent transport with *quantized* time average. The latter corresponds to a phase slippage process which turns out to be a continuous nucleation of *coreless vortices* in *spacetime*. The conductivity tensor for the time average currents is characterized by two topological constants specifying the nucleation process. It is interesting to note that there is a close analogy between QHF and superfluid ^{3}He-A. They have similar order parameters, topological excitations, and vortex nucleation processes.

To determine the magnetotransport of a bilayer system, we need to find the current densities $\mathbf{J}_\mu$ for given electric fields $\mathbf{E}_\mu$ in each layer, where $\mu =\uparrow$ and $\downarrow$ denote the upper and lower layers, respectively. A general configuration $\{\mathbf{E}_\mu\}$ is a sum of identical and opposite fields $(\mathbf{E}_\uparrow, \mathbf{E}_\downarrow) = \mathbf{E}_+(1,1) + \mathbf{E}_-(1,-1)$; $\mathbf{E}_\pm = \frac{1}{2}(\mathbf{E}_\uparrow \pm \mathbf{E}_\downarrow)$. The cases $(\mathbf{E}_+ \neq 0, \mathbf{E}_- = 0)$ and $(\mathbf{E}_+ = 0, \mathbf{E}_- \neq 0)$ will be referred to as the "in-phase" and "out-of-phase" modes. Their corresponding current responses will be denoted by a subscript "+" and "−," respectively. In linear response, it is sufficient to consider these modes separately. The in-phase mode is the usual QH transport, with current response

$$(\mathbf{J}_\uparrow + \mathbf{J}_\downarrow)_+ = \nu \frac{e^2}{h} \hat{\mathbf{B}} \times \mathbf{E}_+, \quad (\mathbf{J}_\uparrow - \mathbf{J}_\downarrow)_+ = 0, \qquad (1)$$

where $\hat{\mathbf{B}}$ is the direction of the magnetic field, taken to be normal to the layers in our discussions. The out-of-phase mode is unique to bilayer systems. For QHFs, we shall show that dc fields $\mathbf{E}_-$ will generate time dependent currents. The sum $(\mathbf{J}_\uparrow + \mathbf{J}_\downarrow)_-$ oscillates with period $T = 2T_0$, where T_0 is the Josephson period $e\Delta V/\hbar$, and ΔV is the voltage that generates $\mathbf{E}_-$. The factor of 2 is a reflection of the double valueness of the pseudo-spin 1/2. The time average current response (denoted by a "bar") can be determined from general argument and are given by

$$\overline{(\mathbf{J}_\uparrow + \mathbf{J}_\downarrow)_-} = n_A \nu \frac{e^2}{h} \hat{\mathbf{B}} \times \mathbf{E}_-,$$
$$\overline{(\mathbf{J}_\uparrow - \mathbf{J}_\downarrow)_-} = n_B \frac{\nu}{2} \frac{e^2}{h} \hat{\mathbf{B}} \times \mathbf{E}_-, \qquad (2)$$

where n_A and n_B are integers. By studying a number of examples consistent with the driving force $\mathbf{E}_-$, we find $n_A = 0$, $n_B = 1$. The sum of Eqs. (1) and (2) then gives

$$\overline{\begin{pmatrix} \mathbf{J}_\uparrow \\ \mathbf{J}_\downarrow \end{pmatrix}} = \nu \frac{e^2}{8h} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \hat{\mathbf{B}} \times \begin{pmatrix} \mathbf{E}_\uparrow \\ \mathbf{E}_\downarrow \end{pmatrix}. \qquad (3)$$

These examples also reveal an interesting oscillatory behavior of the currents. The time average of $(\mathbf{J}_\uparrow + \mathbf{J}_\downarrow)_-$ over the half period T_0 is $\pm\nu(e^2/h)\hat{\mathbf{B}} \times \mathbf{E}_-$, with alternating sign every half period. The time average of $(\mathbf{J}_\uparrow - \mathbf{J}_\downarrow)_-$ is $(\nu/2)(e^2/h)\hat{\mathbf{B}} \times \mathbf{E}_-$ for all half periods.

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To derive these results, we recall that the Hamiltonian of the bilayer system in the weak tunneling limit is [6]

$$H = \int d\mathbf{r}\left[\frac{1}{2m}\hat{\psi}^{\dagger}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^{2}\hat{\psi}-\Delta\hat{m}_1(\mathbf{r})\right]+\frac{1}{2}\int[\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')v(\mathbf{r}-\mathbf{r}')+\hat{m}_3(\mathbf{r})\hat{m}_3(\mathbf{r}')v_1(\mathbf{r}-\mathbf{r}')], \tag{4}$$

where $\mathbf{r}=(x,y)$, $\hat{\psi}_\mu(\mathbf{r})$ creates an electron at the μ layer at $\mathbf{r}$, $\hat{m}_i=\frac{1}{2}\mathrm{Tr}\hat{\psi}^{+}\sigma_i\hat{\psi}$ $(i=1,2,3)$, $\hat{\rho}=\mathrm{Tr}\hat{\psi}^{+}\hat{\psi}$, and both $v, v_1>0$. The pseudo-spin axes $\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3$ are chosen so that the symmetric and antisymmetric states (with energy difference Δ) are represented by $\binom{1}{1}$ and $\binom{1}{-1}$. They have no relations with the real space directions $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$.

We shall consider the (mmm) state in the correlation limit $\Delta \ll e^2/\epsilon_0 l \sim e^2/\epsilon_0 d$, where d is the spacing between layers, and ϵ_0 is the dielectric constant. We begin by expressing their wave functions (in the symmetric gauge) in spinor form

$$\Psi\begin{pmatrix}\mathbf{r}_1,\ldots,\mathbf{r}_N\\ \mu_1,\ldots,\mu_N\end{pmatrix}=e^{-\sum_{i=1}^{N}|z|_i^2/4}\prod_{i=1}^{N}\zeta_{\mu_i}\prod_{i>j}(z_i-z_j)^m, \tag{5}$$

where $z=x+iy$ and

$$\zeta=\begin{pmatrix}u\\ v\end{pmatrix}=|\zeta|\begin{pmatrix}e^{-i\phi/2}\cos\theta/2\\ e^{i\phi/2}\sin\theta/2\end{pmatrix}e^{-i\chi/2} \tag{6}$$

is a spinor along $\hat{l}=\hat{\mathbf{x}}_3\cos\theta+\sin\theta(\hat{\mathbf{x}}_1\cos\phi+\hat{\mathbf{x}}_2\sin\phi)$. Since $v_1>0$, it is energetically more favorable for $\hat{l}$ to lie in the x_1-x_2 plane. Δ will line up $\hat{l}$ along $\hat{\mathbf{x}}_1$. However, in the correlation limit, it remains small compared to the correction energy. The symmetric and antisymmetric states can therefore be regarded as essentially degenerate—a view that we shall adopt in our subsequent discussions for simplicity [7].

If uniform potentials $V_\uparrow$ and $V_\downarrow$ are applied in the upper and lower layers, u and v will evolve as $e^{ieV_\uparrow t/\hbar}$ and $e^{ieV_\downarrow t/\hbar}$ $(e>0)$, meaning

$$\begin{aligned}-\partial_t\phi&=\frac{e}{\hbar}(V_\uparrow-V_\downarrow)\equiv\frac{eV}{\hbar},\\ -\partial_t\chi&=\frac{e}{\hbar}(V_\uparrow+V_\downarrow)\equiv\frac{2eU}{\hbar}.\end{aligned} \tag{7}$$

Equation (7) implies that $\hat{l}$ precesses about $\hat{\mathbf{x}}_3$ with angular frequency $eV/\hbar$. This immediately implies an "unstable" situation if V is different at two points far apart (i.e., $\mathbf{E}_-\neq 0$). The $\hat{l}$ vectors at these points will assume their ground state configurations (lying in the x_1-x_2 plane), but rotate about $\hat{\mathbf{x}}_3$ at different rates. As a result, more and more winding in the spin texture is produced which can only be relieved by phase slippage processes. As we shall see, during phase slippage, $\hat{l}$ will be pulled away from the x_1-x_2 plane every now and then in some region in space. While this costs nucleation energy because of the last term in Eq. (4), it cannot be stopped because its energy cost will eventually be surpassed by the gradient energy generated by $\mathbf{E}_-$.

Nonuniform textures imply spatially varying spinors. However, to stay in the lowest Landau, *ζ must be analytic in z (or a function of ∂_z)*. For simplicity, we shall from now on focus on the "quasihole" case where $\zeta(z)$ is an analytic function of z. In general, ζ can be expanded in powers of z. The simplest example of a nonuniform analytic spinor is $\zeta(z)=\alpha+\beta z$, where ζ reduces to spinors α and β as z approaches 0 and ∞. It also reduces to the usual quasihole excitations if $\alpha\propto\beta$. For arbitrary ζ, Eq. (5) implies that

$$2\mathbf{m}(\mathbf{r})=\rho(\mathbf{r})\hat{l}(\mathbf{r})\,. \tag{8}$$

This is a statement of "full" ferromagnetism and is important for our subsequent discussions.

The relations between density and spin fluctuation can be obtained from the "plasma analog" [8], where one interprets $\sum_{[\mu]}|\Psi([\mathbf{r},\mu])|^2$ as the partition function of a classical 2D plasma with charge m in a background potential $V_b(z)=-|z|^2/2+\ln|\zeta|^2$. This implies that the electron density is $\rho(\mathbf{r})=-(4\pi m)^{-1}\nabla^2V_b=(2\pi m)^{-1}-(4\pi m)^{-1}\nabla^2\ln|\zeta|^2$. Because of the *analyticity* of ζ, the density change can be expressed in terms of a deformation field [9] $\mathbf{u}=-i\zeta^{+}\nabla\zeta/|\zeta|^2$+c.c. as

$$\rho(\mathbf{r})=\frac{1}{2\pi m}-\frac{1}{4\pi m}\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}. \tag{9}$$

It follows from Eq. (6) that [10,11]

$$\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}=-(\hat{\mathbf{z}}\times\nabla)\cdot\nabla\chi+\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}l_\alpha(\hat{\mathbf{z}}\times\nabla)l_\beta\cdot\nabla l_\gamma. \tag{10}$$

The first term vanishes unless χ is a vortex, in which case it is a δ function [12].

The total electric current $\mathbf{J}=\mathbf{J}_\uparrow+\mathbf{J}_\downarrow$ can be obtained by differentiating the number density Eq. (9) and using the continuity equation $(-e)\dot{\rho}=-\nabla\cdot\mathbf{J}$,

$$J_\alpha=\frac{1}{m}\frac{e^2}{h}(\hat{\mathbf{z}}\times\nabla)_\alpha U-\frac{e}{4\pi m}\hat{l}\cdot(\hat{\mathbf{z}}\times\nabla)_\alpha\hat{l}\times\partial_t\hat{l}, \tag{11}$$

where we have made use of Eq. (7). The first term is the usual QH transport, Eq. (1). [Note that $\mathbf{E}_+=-\nabla U$ and that Eq. (5) is derived for $\mathbf{B}=-\hat{\mathbf{z}}$.] The second term is due to the motion of the spin texture, which is generated by $\mathbf{E}_-$. To simplify discussions and formulas, we shall from now on focus on the out-of-phase mode, i.e., $\nabla U=0$, $\nabla V\neq 0$. We shall then drop the first terms in Eqs. (11) and (10).

Consider now the time average of the current in y over a period T, $\overline{(I_\uparrow+I_\downarrow)_y}=T^{-1}\int_0^T dt\int J_y\,dx$. The integral is evaluated for an arbitrary y coordinate, with upper and lower limits x_a and x_b denoting the edges of the sample. From Eq. (11) we have

$$\overline{(I_\uparrow + I_\downarrow)_y} = \frac{-e}{4\pi m}\frac{1}{T}\left[\int_0^T dt \int_{x_a}^{x_b} dx\, \hat{l}\cdot\partial_x\hat{l}\times\partial_t\hat{l}\right]. \quad (12)$$

The quantity in the brackets in Eq. (12) (denoted as A) is the area on a unit sphere S_2 covered by the "spacetime rectangle" Γ with area $T\times L$, $L=(x_b-x_a)$. A non-vanishing A means nucleation of vorticity in spacetime $[\partial_t(\nabla\times\mathbf{u})\neq 0]$.

Equation (12) alone cannot determine $I_{\uparrow y}$ and $I_{\downarrow y}$. From Eqs. (8), (9), and (10), we have

$$2\mathbf{m}(\mathbf{r}) = \frac{1}{2\pi m}\hat{l} - \frac{1}{4\pi m}\partial_x\hat{l}\times\partial_y\hat{l}, \quad (13)$$

$$2\partial_t\mathbf{m} = \frac{1}{2\pi m}\partial_t\hat{l} - \frac{1}{4\pi m}\nabla_a[(\hat{\mathbf{z}}\times\nabla)_\alpha\hat{l}\times\partial_t\hat{l}]. \quad (14)$$

Equation (14) says that spin fluctuations will generate a pseudo-spin current

$$\mathbf{K}_\alpha = \frac{1}{4\pi m}(\hat{\mathbf{z}}\times\nabla)_\alpha\hat{l}\times\partial_t\hat{l}, \quad (15)$$

which is related to the charge currents as $(-e)\hat{\mathbf{x}}_3\cdot\mathbf{K}_\alpha = J_{\uparrow\alpha} - J_{\downarrow\alpha}$. The time average of current difference in y is

$$\overline{(I_\uparrow - I_\downarrow)_y} = \frac{-e}{4\pi m}\frac{1}{T}\left[\int_0^T dt \int_{x_a}^{x_b} dx\, \hat{\mathbf{x}}_3\cdot\partial_x\hat{l}\times\partial_t\hat{l}\right]. \quad (16)$$

The quantity in the brackets (denoted as B) is the area in S_2 covered by Γ projected onto the x_1-x_2 plane.

To evaluate A and B, we consider a rectangular sample (see Fig. 1) where the potentials $(V_\uparrow, V_\downarrow)$ at x_a and x_b are (0,0) and $(V/2, -V/2)$, respectively. This is the "unstable" situation we discussed before, where $\hat{l}$ precesses about $\hat{\mathbf{x}}_3$ with angular frequency $eV/\hbar$ at x_b and remains stationary at x_a [13]. In the steady state, the textural motion must be periodic as $\mathbf{E}_-$ forces a constant rotation on ζ at x_b. Note that $\hat{l}$ at x_b returns to itself after time $T_0 = h/eV$, whereas the period of ζ [and hence the entire texture $\hat{l}(x,y,t)$] is $T=2T_0$. The factor of 2 is due to the double valueness of ζ. An example of the steady state evolution of $\hat{l}$ is shown in the spacetime plot (x-t) in Fig. 1. The evolution is a repetition of the pattern inside the rectangle $T\times L$ ($abb''a''$). Since the boundary of the rectangle maps onto the equator of S_2 twice, we then have $A=-4\pi n_A$, $B=-2\pi n_B$, where n_A and n_B are integers depending on how S_2 is covered. Equations (12) and (16) then give $\overline{(I_\uparrow + I_\downarrow)_y} = n_A(1/m)(e^2/h)(V/2)$, $\overline{(I_\uparrow - I_\downarrow)} = n_B(2m)^{-1}(e^2/h)(V/2)$, which is Eq. (2) since $\mathbf{E}_- = -\frac{1}{2}\nabla(V_\uparrow - V_\downarrow) = -\frac{1}{2}\nabla V$ and $\hat{\mathbf{B}} = -\hat{\mathbf{z}}$.

To determine the topological integers n_A, n_B, we construct explicit examples for the out-of-phase current response. Since we are dealing with rectangular geometries, it is more convenient to use the Landau gauge. The QHF now reads [14]

$$\Psi\begin{pmatrix}[\mathbf{r}]\\ [\mu]\end{pmatrix} = e^{-\sum_i x_i^2/2}\prod_i \zeta(w_i)\prod_{i>j}(w_i - w_j)^m, \quad (17)$$

where $w = e^{\alpha z}$ and α is the ratio of the magnetic length to the sample length in y, which can be taken as 1 without loss of generality. To match the Josephson relations Eq. (7) at the boundaries x_a and x_b (now taken to be $-\infty$ and $+\infty$), we choose a spinor of the form

$$\zeta(w) = \lambda\begin{pmatrix}1\\1\end{pmatrix} + e^{x+iy}\begin{pmatrix}e^{ieVt/2\hbar}\\ e^{-ieVt/2\hbar}\end{pmatrix}, \quad (18)$$

where λ is a constant setting the scale of textural destortion in real space [15]. Its textural evolution in the space-

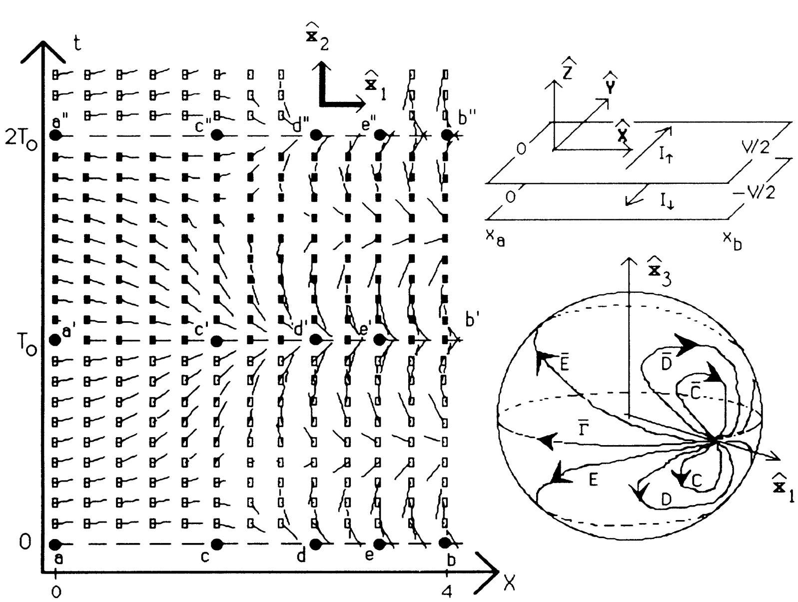

FIG. 1. The $\hat{l}$ vectors at $\mathbf{r}$ are represented by a line emerging from $\mathbf{r}$. A solid (empty) square is placed at $\mathbf{r}$ to indicate that $\hat{l}$ has a positive (negative) x_3 component. The boundary of the rectangle $\Gamma = (abb'a')$ maps onto the equator $\bar{\Gamma}$ on the unit sphere S_2. The lines $\overrightarrow{cc'}$, $\overrightarrow{dd'}$, $\overrightarrow{ee'}$, $\overrightarrow{c'cc''}$, $\overrightarrow{d'd''}$, and $\overrightarrow{e'e''}$ map onto loops $C, D, E, \bar{C}, \bar{D}, \bar{E}$. This covering gives $A=2\pi$, $B=-\pi$ for the lower rectangle $abb'a'$ and $A=-2\pi$, $B=-\pi$ for the upper rectangle $a'b'b''a''$. If the loops $\bar{C}$, $\bar{D}$, and $\bar{E}$ are traversed in directions opposite to those depicted, then $A=2\pi$, $B=\pi$ for $a'b'b''a''$. The textural pattern is calculated from Eq. (18) with $T_0=1$, $y=2$, $\lambda=8$.

time plane (x-t) (for a given y) is shown in Fig. 1. The phase slipping nature of Eq. (18) is obvious in this figure: $\hat{l}$ reduces to a uniform configuration every half period $T_0 = h/eV$. Despite the continuous winding at $x = +\infty$, the precessional angle of $\hat{l}$ about $\hat{\mathbf{x}}_3$ when going from $x = -\infty$ to $+\infty$ never exceeds 2π. [This angle would increase indefinitely if the x_1 and x_2 components of $\hat{l}$ never vanish. The fact that the spin vector of Eq. (18) points to $\pm\hat{\mathbf{x}}_3$ at certain points in the bulk every now and then allows the texture to get rid of whatever amount of twisting accumulated in its x_1 and x_2 components.] The topological areas A and B for each half period can be calculated from Eq. (18) and are independent of λ. A is found to be $\mp 2\pi$, with alternating signs every half period. B is found to be $-\pi$ for all half periods. This means over the full period $T = 2T_0$, $A = 0$ and $B = -2\pi$, corresponding to $n_A = 0$, $n_B = 1$. We have then derived Eq. (3) from this specific example. I have also examined a number of other spinors satisfying the same Josephson boundary condition at infinities. Their time average responses are identical to that of Eq. (18). I therefore think that these values of n_A and n_B have general validity, despite the lack of a general proof which is certainly desirable. On the other hand, spinors of the form Eq. (18) may be realized for energetic reasons. Note that the up and down components of Eq. (18) ($\propto w + \lambda e^{-ieVt/2\hbar}$ and $w + \lambda e^{+ieVt/2\hbar}$) correspond to quasiholes in the upper and lower layers moving in opposite directions (and coalescing into an ordinary quasihole at regular intervals T_0). A general $\zeta(w)$ satisfying the same boundary condition will contain more zeros (i.e., moving quasiholes) in the upper and lower layers than the spinor Eq. (18), and is therefore energetically more costly.

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* Permanent address.

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[5] This is because the "plasma" matrix $(^{mn}_{nm})$ of the (mmn) state vanishes when $m = n$.

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[7] The assumption of degeneracy is not essential for our results as long as Δ acts as a perturbation. For $\Delta \neq 0$, $\mathbf{m}$ (initially along $\hat{\mathbf{x}}_1$) will precess about $\mathbf{h}_{\text{eff}} = \Delta\hat{\mathbf{x}}_1 + eV\hat{\mathbf{x}}_3$ as V is turned on. This will induce a tunneling current $I = -e\partial_t m_3(t) = -(e/\hbar)\Delta m_2(t)$ between the layers. The precession of $\mathbf{m}$ about $\mathbf{h}_{\text{eff}}$ with frequency $\Omega = [(eV)^2 + \Delta^2]^{1/2}$ implies that $I = (\Delta^2|m|/\hbar\Omega)\sin(t\Omega/\hbar)$, which reduces to the result of Wen and Zee in Ref. [3] when $eV \gg \Delta$. Detailed discussions of the effect of tunneling will be given elsewhere.

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[9] When expressed in polar angles, this is exactly the superfluid velocity $\mathbf{v}_s$ of ^{3}He-A. See N. D. Mermin and T. L. Ho, Phys. Rev. Lett. **36**, 594 (1976); T. L. Ho, Phys. Rev. B **18**, 1144 (1978). However, unlike $\mathbf{v}_s$, $\mathbf{u}$ is invariant under time reversal.

[10] This is the vorticity equation of ^{3}He-A. [See Mermin and Ho (Ref. [9]), and Ho (Ref. [9]).] The relation of density and spin fluctuation for polarized QH systems was pointed out by S. L. Sonhi *et al.* [Phys. Rev. B **47**, 16 419 (1993)] in the context of the Chern-Simon theory. It is also derived by the authors in Ref. [2] using projected operator methods. Our derivation here is different from these approaches. It also reveals some rather subtle identities which has significant implications on the energetics of the textural excitations (see Ref. [11]).

[11] Equation (10) can be easily obtained using the identity $\zeta_\mu\zeta_\nu^* = \frac{1}{2}(\delta_{\mu\nu} + \hat{l}\cdot\boldsymbol{\sigma}_{\mu\nu})|\zeta|^2$. For analytic spinors (quasihole type), their textures satisfy $(\hat{\mathbf{z}} \times \nabla)_\alpha \hat{l} = \nabla_\alpha \hat{l} \times \hat{l}$, which implies $\nabla_\alpha l_\beta \nabla_\alpha l_\beta = 2\hat{l}\cdot\partial_x\hat{l} \times \partial_y\hat{l}$. The integral $\frac{1}{2}\int dr(\nabla_\alpha\hat{l})^2$ is therefore a topological constant $4\pi n$ if the texture is uniform at infinity.

[12] A singular vortex can be generated if ζ contains an usual quasihole, i.e., both u and v have a common factor $(z-a)$. Note that the plasma analog is only accurate on length scales larger than the magnetic length. All δ functions revealed by the plasma analog are in fact Gaussians.

[13] The situation here is identical to ^{3}He-A in a heat flow, where the texture is continuously being wound up by a chemical potential gradient. The winding is undone by a periodic nucleation of coreless vortices, which is relieved by a periodic texture motion. See T. L. Ho, Phys. Rev. Lett. **41**, 1473 (1978).

[14] Equations (8)–(10) can also be derived from Eq. (17). The plasma analog can be generalized easily to the Landau gauge by realizing that density of the plasma $\rho(w)$ is related to the electron density $\rho(z)$ by the Jacobian $|dw/dz|^2$.

[15] It follows from Eq. (13) that for spinor Eq. (18), $\int m_3 = 0$. This is consistent with the degenerate (or zero tunneling) limit $\Delta \rightarrow 0$ that we have taken.