Tunneling versus Chaos in the Kicked Harper Model

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We study the interplay between tunneling and chaos in a quantum system which classically would be weakly chaotic. We show that the tunneling rate between two stable islands is exponential and regular when the characteristic size of the chaotic region separating the islands is much larger or much smaller than Planck's constant. When the chaotic region and Planck's constant are of the same size the tunneling rate is shown to be irregular. This result is obtained by means of a numerical analysis of the quantum kicked Harper model, but we argue this to be a generic effect of classical chaos on tunneling.

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The search for the quantum properties of those systems which are chaotic within the theoretical framework of classical mechanics is an active field of investigation [1]. On the other hand, tunneling is a typically quantum mechanical effect and consequently it would be of extreme importance to assess what influence if any classical chaos might have on it. In the past few years there has been an increasing interest in this problem [2-6]. The attention of most researchers focused on a particle moving within a double-well potential under the action of a coherent perturbation. It has been conjectured that chaos might enhance the tunneling rate by several orders of magnitude [3]. However, the more recent theoretical analysis of some authors [4,5] shows that this enhancement of the tunneling rate (or, in some cases, the opposite effect of quenching) can be explained by adopting the relatively simple scheme of a time periodic force acting on a quantum mechanical doublet of states. Thus the coincidence of enhanced tunneling and enhanced chaos [3] cannot be regarded as being a direct effect of chaos on tunneling, but rather as a consequence of a perturbation, which, while increasing the size of the chaotic region, also affects the dynamics of the tunneling doublet.

More relevant to the discovery of a real influence of chaos on tunneling seem to be some results reviewed by Bohigas *et al.* [6] and the paper by Uttermann *et al.* [7]. The latter analysis shows that the dependence of the tunneling rate on the nonlinearity strength significantly departs from the unperturbed case when the size of the tunneling wave packet becomes comparable to that of the stable islands of the periodically perturbed double-well potential.

The purpose of the present Letter is that of giving further support to properties of this kind using the so-called kicked Harper model (KHM). The KHM, widely studied in the literature of quantum chaos [8], is the quantum mechanical version of the following classical area preserving mapping:

$$\begin{aligned} q_{n+1} &= q_n - 2\pi\epsilon \sin(2\pi p_n) , \\ p_{n+1} &= p_n + 2\pi\epsilon \sin(2\pi q_{n+1}) . \end{aligned} \tag{1}$$

This system is periodic with respect to both the coordinate q and the momentum p, thereby making it possible for us to describe the phase-space structure of it by means of a single unit square, or *tile*. This is illustrated by Fig. 1 which shows the structure of the phase space for three different values of the parameter ϵ, the nonlinearity strength. We see that at the smallest value of the nonlinearity strength the dynamics is dominated by regular orbits enclosed by separatrices [Fig. 1(a)]. At larger values of the nonlinearity strength the motion in the close vicinity of the separatrix becomes distinctly chaotic [1(b)], and upon further increase the size of the chaotic region increases [1(c)].

To study the quantum version of KHM one needs to quantize map (1) over a torus containing an integer number of tiles. The quantization prescription we adopt [9] rests on imposing periodic boundary conditions on both the observables $\hat{q}$ and $\hat{p}$ (i.e., $q + q_0 = q$ and $p + p_0 = p$, with q_0 and p_0 integer numbers), and leads to a finite Hilbert space. Note that the above procedure is allowed only in the case where the Planck constant h is a rational number. Indeed, if M is the finite number of the basis states used, h must fulfill $h = \frac{q_0 p_0}{M}$, where

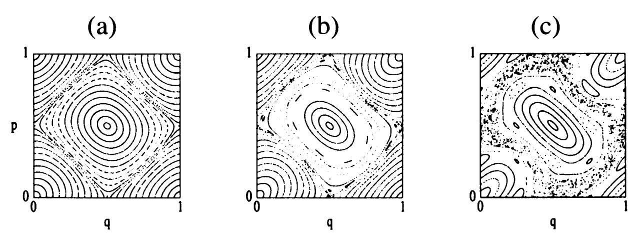

FIG. 1. The classical phase-space structure for different ϵ. The values of the nonlinear parameter are (a) $\epsilon = 0.01$, (b) $\epsilon = 0.025$, and (c) $\epsilon = 0.04$.

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q_0 and p_0 are the integer numbers that specify the size of the torus in the q and p directions. It is important to point out that the classical system depends only on the parameters q_0, p_0, and ϵ which are independent of the quantum parameter M.

The quantum KHM is described by the quantum counterpart of the Hamiltonian generating map (1)

$$H = \epsilon \cos(2\pi \hat{p}) + \epsilon \cos(2\pi \hat{q}) \delta_1(t), \quad (2)$$

where $\delta_1(t)$ is a periodic delta function of unit period. The Floquet operator, namely, the unitary operator which evolves the wave function for one period of the external perturbation, is

$$\hat{U} = \exp\left[-i \frac{\epsilon}{\hbar} \cos(2\pi \hat{q})\right] \exp\left[-i \frac{\epsilon}{\hbar} \cos(2\pi \hat{p})\right]. \quad (3)$$

We call $|m\rangle$ the eigenstates of the $\hat{p}$ operator. On the basis of these states the Floquet operator (3) reads

$$U_{m,m_1} = e^{-i \frac{2\pi\epsilon M}{q_0 p_0} \cos\left(2\pi \frac{p_0 m_1}{M}\right)} \times \frac{1}{M} \sum_{k=0}^{M-1} e^{2\pi i \frac{k(m_1 - m)}{M}} e^{-2\pi i \frac{\epsilon M}{q_0 p_0} \cos\left(2\pi \frac{q_0 k}{M}\right)}. \quad (4)$$

Notice that the matrix element (4) is left unchanged by translations of an integer number of tiles in the p (q) direction. Thus it is possible to further reduce the dimension of the Hilbert space by exploiting the corresponding symmetry. Let us focus on the two-tile case, i.e., $q_0 = 1$, $p_0 = 2$, which is the object of the numerical investigation of this Letter. It is easy to realize that the definition of operator $\hat{S}$, which corresponds to the shift by one unit in the p direction, has to be conveniently adapted to whether M is even or odd. If M is even, defining N as $M/2$ we get

$$\begin{aligned} \hat{S}|m\rangle &= |m + N\rangle, \quad m = 0, \ldots, N-1, \\ \hat{S}|m\rangle &= |m - N\rangle, \quad m = N, \ldots, M-1. \end{aligned} \quad (5)$$

If M is odd, we define N as $(M-1)/2$, and we get

$$\begin{aligned} \hat{S}|m\rangle &= |m + N + 1\rangle, \quad m = 0, \ldots, N-1, \\ \hat{S}|m\rangle &= |m\rangle, \quad m = N, \\ \hat{S}|m\rangle &= |m - N - 1\rangle, \quad m = N+1, \ldots, M-1. \end{aligned} \quad (6)$$

The eigenstates of the tile-shift symmetry $\hat{S}$ are usually given by the symmetric and antisymmetric combination of states $|m\rangle$ and $\hat{S}|m\rangle$. Using these states as the basis set we divide the operator $\hat{U}$ into two blocks which can be separately diagonalized. For the group of symmetric states the explicit expression for the matrix elements of the operator $\hat{U}$ coincides with the expression (4) for the one-tile torus, but with a number of states equal to N. The corresponding expression for the group of antisymmetric states is slightly different, and it also depends on whether M is even or odd. We give only the even M case:

$$U_{m,m_1} = e^{-i2\pi\epsilon N \cos\left(2\pi \frac{m_1}{N}\right)} \times \frac{1}{N} \sum_{k=0}^{N-1} e^{i(m_1 - m)\left[2\pi \frac{k}{N} + \varphi\right]} e^{-2\pi i \epsilon N \cos\left(2\pi \frac{k}{N} + \varphi\right)}. \quad (7)$$

The difference with the symmetric case is given by the presence of the phase $\varphi = \frac{2\pi}{M}$ which is essentially coincident with the Planck constant h ($h = \frac{2}{M}$). This leads us to conclude that, in the case of a small h, the states that diagonalize the symmetric and antisymmetric blocks must be almost the same, and that both must virtually coincide with the eigenstates of the single-tile case.

Let us now proceed to the numerical analysis of the system. In Fig. 2 we consider the single-tile model and we show the phase-space representation of some eigenstates for the case $M = 30$ and $\epsilon = 0.04$, by means of the Husimi distribution properly extended to our case [10]. Note that this quantum condition corresponds to the classical phase space shown in Fig. 1(c). A widely accepted property of a classically chaotic quantum system is that there exist eigenfunctions peaking on the invariant classical tori, thereby giving rise to localized states. We find that the state illustrated by Fig. 2(a), and located in the middle of the deterministic island, fits this expectation. On the other hand, there also exist states which are essentially located in the chaotic region, as shown in Figs. 2(b) and 2(c). Thus we have available a criterion to divide the Floquet states into *regular* and *irregular* states, according to whether, within the Husimi representation, the corresponding distributions belong to the regular or the irregular portion of the classical phase space.

In the case of a single tile this classification of the Floquet states makes it possible to predict qualitatively the quantum evolution of a wave packet. In fact an initial state entirely located in the regular regions is expressed as a linear superposition of only regular eigenstates, and for this reason does not significantly move out of this region. Similarly, a state initially located in the stochastic web will never significantly invade the regular region.

The tunneling phenomenon requires at least two tiles. In this case we proceed as follows. We diagonalize the

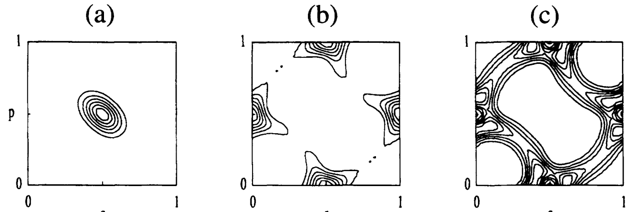

FIG. 2. Contour plots of the phase-space representation of 3 out of the 30 eigenstates of the Floquet operator in the case $M = 30$, $\epsilon = 0.04$.

Floquet operator and look for the doublets corresponding to states located in the island placed at the center of the tile. Of course these doublets consist of symmetric and antisymmetric states with respect to the tile shift. For each doublet, a state localized in one tile is obtained by the symmetric superposition of these two states. Among the doublets we choose that corresponding to the localized state whose overlap with a minimum uncertainty Gaussian wave packet centered on the elliptic fixed point is maximum. Typically the value of this overlap is about 90%. This is the *main doublet*, whose quasienergy splitting ΔE determines the tunneling rate. Within the theoretical framework of an Einstein-Brillouin-Keller semiclassical quantization procedure this should correspond to ensuring that upon change of h the torus closest to the elliptic point is always selected. The phase-space representation of the two states of the main doublet can be obtained using Fig. 2(a). This depicts half of phase space. With $q_0 = 1$ and $p_0 = 2$, the complete phase space of the two states consists of two tiles, each of them with a contour plot similar to that of Fig. 2(a). The phase-space representation of the symmetric state of the main doublet coincides with the resulting two-tile picture, whereas that of the antisymmetric state would be slightly different from it due to the mathematical properties outlined above; see Eq. (7).

The influence of classical chaos on tunneling is proved by numerical evaluation of the quasienergy splitting of the main doublet (proportional to the tunneling rate) as a function of M at different values of ϵ. At $\epsilon = 0.025$ we find (see Fig. 3) that for both small and large M's ΔE undergoes oscillations around a mean value resulting in a regular exponential decrease upon increase of M. In between these two *regular* regions an intermediate region appears where the tunneling rate is proven to be a random function of M. To make the physics behind this figure more transparent, we must remark that, due to the symmetry properties discussed earlier, the even and odd M cases are expected to result in different expressions for the tunneling rate. The dotted lines of Fig. 3 are guides for the eye, showing that the tunneling rate would be an exactly exponential function of M if only even or only odd M were considered.

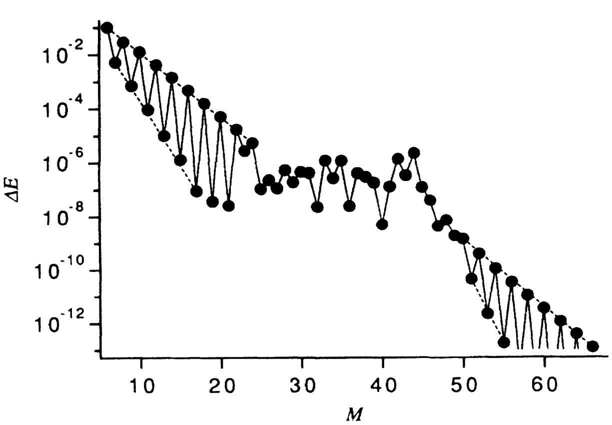

FIG. 3. Energy splitting of the main doublet versus M; $\epsilon = 0.025$.

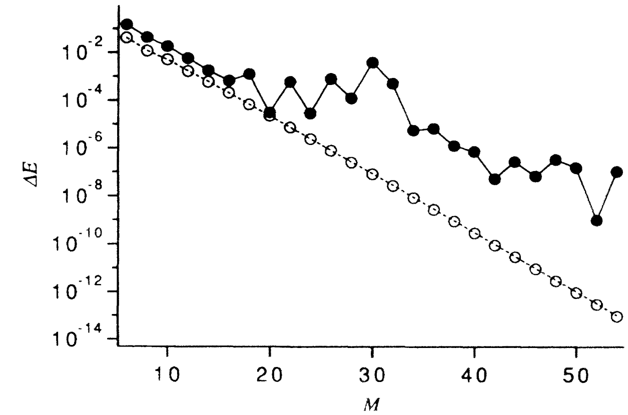

FIG. 4. Energy splitting of the main doublet versus the even M's; $\epsilon = 0.01$ (○) and $\epsilon = 0.04$ (•). The dashed line denotes the result obtained by numerically solving the integrable case for $\epsilon = 0.01$. The solid line is a guide for the eye.

The numerical results obtained show that the dependence of this behavior on the nonlinear strength is as follows. At extremely weak values of ϵ the size of the intermediate irregular region is negligible. Upon increasing ϵ it becomes more significant and more extended. Because of space limitations, here we only report, in Fig. 4, the result corresponding to a small and a large ϵ. The line denoted by open circles corresponds to the almost nonchaotic classical case of Fig. 1(a), where the stochastic region is virtually invisible. We see that the rate of tunneling dependence coincides with that of the integrable system denoted by the dashed line in this figure (we obtained this result numerically from the nonkicked Harper model). The line denoted by filled circles shows the effect of a large ϵ. Note that in this case the regular region corresponding to the large values of M is not shown. We expect that in this physical condition this regular region should appear at such large values of M, $M > 50$, and consequently in correspondence with such a weak quasienergy split, as to be totally hidden by the numerical roundoff errors.

In conclusion, the numerical results lead us to the following, very transparent, physical interpretation. The deterministic island has a characteristic size. If the Planck constant is much smaller than the size of the island, the logarithmic plot leads to a straight line with a certain rate (see Fig. 3): the tunneling rate increases exponentially with increasing h. This result fits the usual semiclassical prediction for the tunneling rate of one-dimensional systems expressed by the exponential-like formula $\Delta E = Ae^{-S/h}$, where S is the classical action associated with the tunneling. Note that the same law holds true also for time independent quasi-integrable systems [11], and surprisingly enough, as shown here for moderate nonlinearity, also in the case of the quantum KHM. We call this region of the Planck constant the *semiclassical region* (SR). On the other hand, when the Planck constant h is large enough we recover an exponential, and monotonic, dependence of the tunneling rate on the Planck constant. This is so because h is so large as to make the system insensitive to the classical chaotic structure [12,13], and consequently indistinguishable from the integrable case, as confirmed by Fig. 4 showing that the slope of the tunneling rate is close to that of the integrable case. We call this region the *quantum region* (QR). We then find an *intermediate irregular region* (IIR), where the tunneling rate is an irregular and nonmonotonic function of h, extremely sensitive to the value of the two parameters of the quantum KHM. Note that the numerical analysis proves this behavior to be determined essentially by the properties of the main doublet, thereby ruling out the influence of a third state as in the case discussed by Bohigas *et al.* [6].

Thus we are led to predict that if we increase the intensity of the nonlinear strength the size of the deterministic islands shrinks, and the size of the chaotic sea is enhanced. As a consequence, the border between the IIR and the SR moves towards the region of small Planck constants h. On the other hand, since upon increasing nonlinearity the size of the stochastic sea increases, the border between the QR and the IIR tends to move towards the large values of the Planck constant. As an effect of this the size of the IIR is expected to increase upon increasing the strength of the nonlinearity. This is fully confirmed by the numerical results.

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[1] See, for example, L. E. Reichl, *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, New York, 1992); or M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

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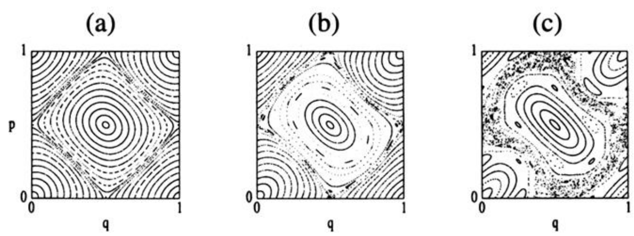

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