

Cosmological No-Hair Theorem

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A generalization of Price's theorem is given for application to inflationary cosmologies. Namely, we show that on a Schwarzschild-de Sitter background there are no static solutions to the wave or gravitational perturbation equations for modes with angular momentum greater than their intrinsic spin.

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A most intriguing feature of gravitational collapse is the way in which the final state seems to be characterized by only a few parameters. This is the phenomenon that Wheeler described as the loss of hair by the black hole [1] and forms the content of the no-hair conjecture.

Implicit in the no-hair conjecture is the idea that gravitational collapse reaches a stationary state. This is supported by a combination of numerical and perturbative calculations. The part of the no-hair conjecture that has been proved rigorously consists of uniqueness theorems for stationary black hole solutions.

Early work on the nonrotating case can be further subdivided into global results [2] and the results summarized by Price's theorem [3]: The only static solutions to the pure massless wave equations with spin $s = 0, \frac{1}{2}, 1$ or to the gravitational perturbation equations on a spherically symmetrical black hole background have angular momentum less than s .

The implication is that all of the other modes are radiated away during the collapse and that the event horizon is characterized by the constants of integration for the remaining modes. For pure gravity these are the mass perturbation and the angular momentum of the hole (the initial angular momentum being zero).

In this Letter we will present a generalization of Price's theorem for black holes in asymptotically de Sitter spacetimes. These black holes have been of interest recently because there are situations in which they form naked singularities [4-7]. However, the overriding reason for looking at no-hair theorems with a cosmological constant is because of their role in inflationary models of the early Universe.

The cosmic no-hair conjecture states that in the presence of a cosmological constant the universe evolves into a de Sitter universe [8,9]. This would imply that inflation is a natural phenomenon that can explain the isotropy and homogeneity seen in the Universe on large scales. In point of fact this simple version of the cosmic no-hair conjecture is violated simply by generalizing the Schwarzschild spacetime to include a cosmological constant [10]. Valid cosmological no-hair theorems can be obtained, however, for the homogeneous universe [11,12] or else in the vicinity of an infinitely future extendable world line [13].

It is also possible to derive global uniqueness theorems for stationary solutions to the Einstein equations for gravity with a cosmological constant [14,15]. What has not been achieved so far is a theorem of this kind with weak enough conditions at infinity that would allow black holes and generalize the uniqueness theorems for static solutions in asymptotically flat spacetimes [2].

If the earliest stages of our Universe were very chaotic then parts would have been collapsing under the influence of gravity while others were expanding [16-19]. It would be desirable to understand how the formation of black holes would affect the inflation going on around early episodes of gravitational collapse.

We will prove the generalization of Price's theorem first and then discuss the approach to de Sitter space outside the hole: The only static solutions to the pure massless wave equations with spin $s = 0, \frac{1}{2}, 1$ or to the gravitational perturbation equations on a Schwarzschild-de Sitter background have angular momentum less than s .

The proof requires the explicit forms of the wave equations on the Schwarzschild-de Sitter background. The background metric is a vacuum solution to the Einstein equations with a positive cosmological constant Λ ,

$$ds^2 = -r^{-2}\Delta dt^2 + r^2\Delta^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$\Delta = r^2 - 2Mr - \frac{1}{3}\Lambda r^4. \quad (2)$$

There are two horizons where Δ vanishes, a black hole event horizon at $r = r_2$ and a cosmological horizon at $r = r_1$. The metric approaches de Sitter space as $r \rightarrow \infty$.

Fortuitously, all of the relevant equations on this background now exist in the literature [4,7]. We begin with the pure spin s wave equations. In $SL(2, C)$ spinor notation,

$$\nabla_{AA'}\Psi^{AB\dots C} = 0. \quad (3)$$

The field has $2s$ symmetrized indices and therefore $2s + 1$ independent components, which we label Ψ_n , $n = -s, \dots, s$.

We take

$$\Psi_n = r^{-s} R_n(j, \omega; r) S_n(j, m; \theta) e^{(im\phi - i\omega t)}, \quad (4)$$

where the angular functions are spin n eigenfunctions of

$$\left[\mathcal{D}_{-n/2}^\dagger \Delta \mathcal{D}_{n/2} + 2(2n-1)i\omega r + 2(2n-1)(n-1) \left(Mr^{-1} - \frac{1}{3} \Lambda r^2 \right) \right] R_n = \lambda_n R_n \quad (5)$$

for $1-s \leq n \leq s$ and

$$\left[\mathcal{D}_{n/2} \Delta \mathcal{D}_{-n/2}^\dagger - 2(2n+1)i\omega r + 2(2n+1)(n+1) \left(Mr^{-1} - \frac{1}{3} \Lambda r^2 \right) \right] R_n = \lambda_{-n} R_n \quad (6)$$

for $-s \leq n \leq s-1$, with $\lambda_n = (j+n)(j-n+1)$. For $s=0$, which is excluded by the inequalities,

$$\left(\mathcal{D}_0 \Delta \mathcal{D}_0^\dagger - 2i\omega r \right) R_0 = \lambda_0 R_0. \quad (7)$$

The operator \mathcal{D}_n is a radial derivative,

$$D_n = \partial r - \frac{i\omega r^2}{\Delta} + n \frac{\Delta'}{\Delta}. \quad (8)$$

[In Kerr-de Sitter the mass term on the left of Eqs. (5) and (6) mixes radial and angular modes. Therefore the examples quoted in Ref. [7] were all cases where this term vanished. We have also shifted the subscript on Ψ by s .]

In the static case we have $\omega=0$. The equations have regular singular points at both horizons $r=r_1$ and $r=r_2$ where Δ vanishes. At these singular points $R_n = O(\Delta^{\pm n/2})$.

Suppose that Eq. (5) has a solution that is regular at both $r=r_1$ and $r=r_2$. If we multiply the equation by R_n^\dagger and integrate between the two horizons then

$$\int_{r_2}^{r_1} \left\{ \Delta (\mathcal{D}_{n/2} R_n)^\dagger (\mathcal{D}_{n/2} R_n) + V(r) R_n^\dagger R_n \right\} dr - [\Delta R_n^\dagger \mathcal{D}_{n/2} R_n]_{r_2}^{r_1} = 0, \quad (9)$$

where

$$V(r) = (2n-1)(n-1)(r^{-2}\Delta + \Lambda r^2) + \epsilon(j, n) \quad (10)$$

and

$$\epsilon(j, n) = \lambda_n - (2n-1)(n-1). \quad (11)$$

The boundary term vanishes for the regular solutions. If $j \geq s > 0$ then $\lambda_n \geq s+n$ and $V(r) \geq 0$ for $n=0, \frac{1}{2}, 1$. If $s=0$ then $V(r) = j(j+1) \geq 0$. These are only consistent if R_n vanishes identically and therefore there are no solutions for $j \geq s$ and $n \geq 0$. The same conclusion follows for negative n from Eq. (6). This is the result that was required.

the Laplacian on the sphere with eigenvalues $j(j+1)$ and half integer angular momenta j .

The equations for the radial functions on a Kerr-de Sitter background were analyzed recently in Ref. [7]. In the Schwarzschild-de Sitter limit they are

If $s=1$, for example, then only Ψ_0 can have a static value. The solution for the radial modes with $j=0$ is $R_0 = Q/r$, where Q is a constant. This represents the radial electric field of a point charge.

The perturbed Einstein equations are different from the pure spin 2 wave equation. We follow the discussion in Ref. [20], and decompose the metric perturbations into polar and axial perturbations depending on their behavior under $\phi \rightarrow -\phi$. For the polar perturbations

$$\delta g_{rr} = 2r^2 \Delta^{-1} L(r) P_l(\theta), \quad (12)$$

$$\delta g_{\theta\theta} = 2r^2 [T(r) + U(r) \partial_\theta \partial_\theta] P_l(\theta), \quad (13)$$

$$\delta g_{\phi\phi} = 2r^2 \sin^2 \theta [T(r) + U(r) \cot \theta \partial_\theta] P_l(\theta). \quad (14)$$

All of the functions $T(r)$, $L(r)$, and $U(r)$ are related by the Einstein equations to a single function,

$$Z_+(r) - rU(r) - r^2(\lambda_2 r + 6M)^{-1} [2L(r) + \lambda_2 U(r)]. \quad (15)$$

For the axial perturbations we set

$$\delta g_{r\phi} = r^2 \Delta^{-1} A(r) \sin \theta \partial_\theta P_l(\theta), \quad (16)$$

$$\delta g_{\theta\phi} = r^2 \Delta^{-1} B(r) (\sin \theta \partial_\theta^2 - \cos \theta \partial_\theta) P_l(\theta), \quad (17)$$

and define $Z_-(r) = [A(r) - B(r)'] / r$. Again, $A(r)$ and $B(r)$ can be found given $Z_-(r)$.

The perturbed Einstein equations for a charged black hole de Sitter background were derived in Ref. [4]. With the charge set to zero these reduce to

$$-\frac{d^2 Z_\pm}{dr^{*2}} + V_\pm(r) Z_\pm = \omega^2 Z_\pm, \quad (18)$$

where $dr^* = r^2 dr / \Delta$. The potentials are given by

$$V_\pm(r) = \pm 6M \partial_{r^*} f + 36M^2 f^2 + \lambda_2(\lambda_2 + 2)f. \quad (19)$$

where

$$f = \frac{\Delta}{(\lambda_2 r + 6M)r^3}. \quad (20)$$

The equations are identical in form to the equations that are obtained for $\Lambda = 0$. The cosmological constant appears only in Δ .

It is possible to transform these equations into a similar form to the previous set. For $\omega = 0$ define

$$R_{\pm} = \frac{r^3}{\Delta} [V_{\pm}(r)Z_{\pm} + W_{\pm}(r)\partial_r \cdot Z_{\pm}] \quad (21)$$

with

$$W_{\pm}(r) = -\partial_{r^*} \ln f \mp 6Mf. \quad (22)$$

The equations for R_{\pm} are then

$$(\mathcal{D}_{-1}^{\dagger} \Delta \mathcal{D}_1 - 2\Lambda r^2)R_+ = \lambda_2 R_+, \quad (23)$$

$$(\mathcal{D}_{-1} \Delta \mathcal{D}_1^{\dagger} - 2\Lambda r^2)R_- = \lambda_2 R_-. \quad (24)$$

These are also the equations for the $n = \pm 2$ components of the Weyl tensor that were derived in Ref. [7]. Proceeding as before, we find that there are no solutions for R_{\pm} that are regular on both horizons apart from $R_{\pm} = 0$. Setting $R_{\pm} = 0$ [see (21)] leads to regular solutions for Z_{\pm} only in the case where $j = 1$ and logarithmically divergent solutions (sufficient for regular L and U) when $j = 0$. Therefore the theorem is proven.

The way in which the spacetime around a collapsing star settles down to the static spacetime is somewhat different from the situation in flat space. The Penrose diagram in Fig. 1 shows the collapse of a spherical star to form a black hole. The metric has been continued beyond each horizon by introducing Kruskal coordinates. Null coordinates are defined by $u = t - r^*$ and $v = t + r^*$, where $dr^* = r^2 dr/\Delta$, and the Kruskal coordinates are

$$U_1 = -e^{-\kappa_1 u} \quad \text{and} \quad V_1 = e^{\kappa_1 v}, \quad (25)$$

$$U_2 = -e^{-\kappa_2 u} \quad \text{and} \quad V_2 = e^{\kappa_2 v}, \quad (26)$$

where κ_2 is the surface gravity of the event horizon.

The values of various field components can be taken to be given on the stellar surface and then will propagate into the exterior spacetime. The surface of the star begins to collapse at $u = u_0$ and lies close to a null surface $v = v_0$ as it approaches the event horizon. In this time-dependent problem we let $\Phi(j, m, n; r, t)$ be a field component with fixed angular eigenvalues j and m .

Each mode of the field should remain analytic in the Kruskal coordinates, and therefore on the surface of the star;

$$\Phi \sim \psi_0 + \psi_1 e^{-\kappa_2 u}. \quad (27)$$

The only modes which have ψ_0 nonzero are those with $j < s$. In flat space the constant modes cannot reach

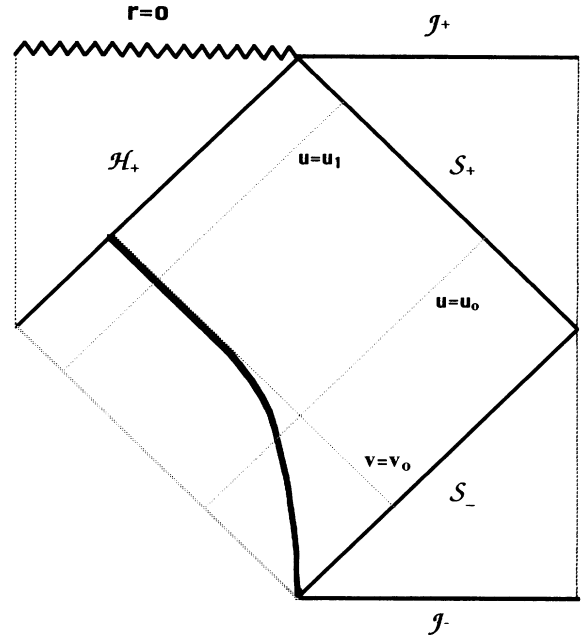


FIG. 1. Penrose diagram for a star collapsing in a de Sitter universe. The collapse starts at retarded time u_0 and the data approach the asymptotic form by u_1 . The diagram extends beyond the edges of the figure.

infinity. In de Sitter space the constant modes propagate all the way to the cosmological horizon.

The surface of the star and the past cosmological horizon S_- form the initial data surfaces. If the star is removed, the complete black hole de Sitter spacetime has a past event horizon \mathcal{H}_- as well as a future event horizon. Close to the horizons the radial modes with frequency ω become plane waves $R_n \sim r^{-1} e^{\pm i\omega r^*}$. We can define separate reflection and transmission amplitudes $R(\omega)$ and $T(\omega)$ for modes which propagate from the past event horizon, denoted by \rightarrow and for modes which propagate from the past cosmological horizon, denoted by \leftarrow .

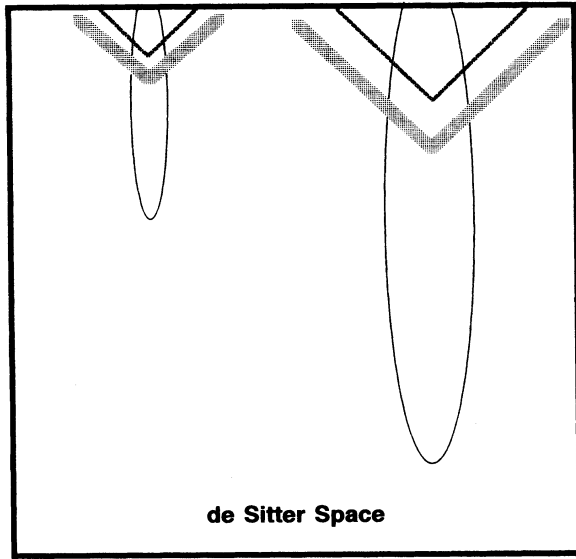
The initial data correspond to specifying two functions $F(u)$ and $G(v)$ on \mathcal{H}_- and S_- . These are analytic in U_2 and V_1^{-1} , the Kruskal coordinates that vanish near \mathcal{H}_+ and S_+ , respectively,

$$F(u) \sim F_0 + F_1 e^{-\kappa_2 u} \quad \text{as } u \rightarrow \infty, \quad (28)$$

$$G(v) \sim G_0 + G_1 e^{-\kappa_1 v} \quad \text{as } v \rightarrow \infty. \quad (29)$$

The modes that are represented by these expansions have frequencies $-i\kappa_2$ and $-i\kappa_1$ as well as zero. Therefore after transmission to the future horizons they become

$$r\Phi(\infty, v) \sim \left[1 + \overrightarrow{R(0)}\right]F_0 + \overrightarrow{T(0)}G_0 + \overrightarrow{R(-i\kappa_2)}F_1 e^{-\kappa_2 v} + \overleftarrow{T(i\kappa_1)}G_1 e^{-\kappa_1 v}, \quad (30)$$



KEY :

outgoing radiation

horizons \mathcal{H}_i

FIG. 2. Penrose diagram of de Sitter spacetime with many distinct collapsing lumps and their radiation fields. Most observers see a universe that approaches the de Sitter universe asymptotically.

$$r\Phi(u, \infty) \sim \left[1 + \overleftarrow{R(0)}\right]G_0 + \overrightarrow{T(0)}F_0 + \overleftarrow{R(-i\kappa_1)}G_1 e^{-\kappa_1 u} + \overrightarrow{T(i\kappa_2)}F e^{-\kappa_2 u}. \quad (31)$$

In the asymptotically flat case the corresponding transmission amplitudes are of order ω^{j+1} for small omega and the constant modes are trapped. The present case is much more similar to the scattering in the interior region of a charged black hole [20], from which we can deduce that

$$\overleftarrow{T(0)} \neq 0, \quad \overrightarrow{T(0)} \neq 0, \quad \overleftarrow{T(-i\kappa_1)} = \overrightarrow{T(-i\kappa_2)} = 0. \quad (32)$$

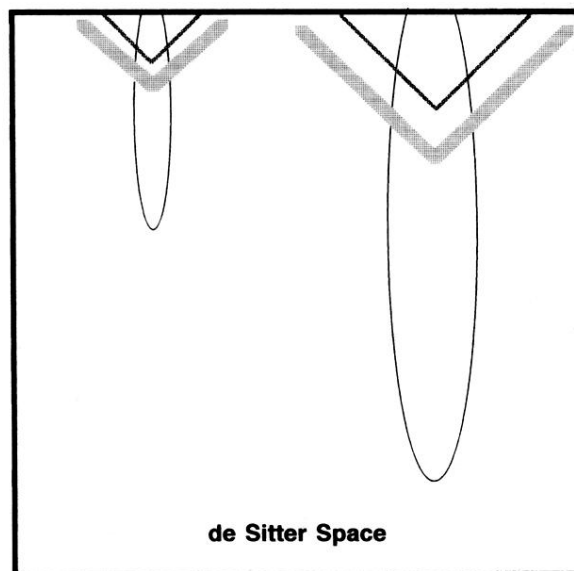
In other words, the constant modes are transmitted to the cosmological horizon but the exponentially decaying modes are not (compare Ref. [21]).

Finally, the radiation emitted during the collapse of the star propagates through the future cosmological horizon and out to infinity. This radiation lies principally between two retarded times u_0 and u_1 . In the vicinity of future infinity the r coordinate is timelike. It can be related to the Robertson-Walker ($k = 1$) cosmological time coordinate τ and radius χ by $r = \alpha \cosh(\tau/\alpha) \sin \chi$ where $\alpha^2 = 3/\Lambda$. Asymptotically, the modes are bounded by $1/r$ and therefore the amplitudes seen by an observer at fixed χ decrease exponentially in τ with a decay time scale α .

In inflationary models the cosmological constant is present for only a limited period but numerical studies of spherically symmetric and axisymmetric models [18,19] indicate that there is enough time for massive inhomogeneities to collapse and form stable black holes before the inflation comes to an end. Thus it seems possible to have a picture of the early Universe in which many inhomogeneities separated by distances comparable to the cosmological horizon scale were collapsing into black holes while other regions of the Universe were inflating [16]. Radiation from the collapsing material would mostly have been damped away by the exponential expansion in the surrounding spacetime before the end of the inflationary period (Fig. 2). Observers outside of the black holes whose world lines extended far into the future would have seen a universe that increasingly appeared to be de Sitter.

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