PHYSICAL REVIEW
LETTERS

VOLUME 73 | 1 AUGUST 1994 | NUMBER 5

There Are No Causality Problems for Fermi's Two-Atom System

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(Received 2 March 1994)

A repeatedly discussed gedanken experiment, proposed by Fermi to check Einstein causality, is reconsidered. It is shown that, contrary to a recent statement made by Hegerfeldt, there appears no causality paradox in a proper theoretical description of the experiment.

PACS numbers: 03.65.Bz

In a recent Letter [1] Hegerfeldt discusses a gedanken experiment proposed by Fermi to determine the speed by which causal influences propagate. He argues that the theoretical description of this experiment in terms of transition probabilities leads to results which are in conflict with the existence of a maximal propagation speed c. Hegerfeldt suggests that the difficulties might disappear if one drops an implicit assumption about the preparability of states with certain specific localization properties [points (a) to (c) of his conclusions]. However, he does not settle the question whether the theory complies with Einstein causality.

In this Letter we would like to set forth that there are no difficulties with Einstein causality in the theoretical setting of relativistic quantum field theory (RQFT). First, we will explain why the transition probability test consider by Hegerfeldt is not adequate for a thorough discussion of causal effects: what is required is a comparison of expectation values. Second, we will show that the points indicated by Hegerfeldt as possible loopholes to evade causality problems have to be taken seriously indeed and require a more careful analysis. Taking these facts fully into account, we arrive at the conclusion that there is no conflict between the gedanken experiment of Fermi and the theoretical predictions of RQFT.

The experimental setup envisaged by Fermi to determine the propagation speed c of causal influences can be described as follows (cf. [2] and, for further references, [1]): one should prepare a state consisting of two atoms which are localized in disjoint regions separated by a distance R. One atom should be in its ground state, the other one in an excited state. If causal influences propagate with maximal velocity c one should not observe any impact of the excited atom on the atom in the ground state (e.g., by an emitted photon) within the time interval $0 < t < R/c$. Such events should be observed only at later times.

It is of importance that the atoms in this experiment have well defined localization properties. Hence in a theoretical discussion of the setup one first has to define in precise terms what one means by the statement that some physical state S (e.g., the state considered by Fermi, consisting of two atoms) looks at time $t = 0$, say, inside a region $\mathcal{R}$ like a given state G (e.g., like an atom in its ground state) [3]. From the point of view of physics the appropriate definition seems to be the following one: it is impossible to distinguish S from G by any measurement M which one performs at the given time in the region $\mathcal{R}$. In the theoretical setting this amounts to the requirement that the expectation values of all operators (observables) O_M corresponding to these measurements have to coincide. Hence if ψ_S, ψ_G denote the Hilbert space vectors representing, respectively, S and G it must hold that

$$\langle \psi_S | O_M | \psi_S \rangle = \langle \psi_G | O_M | \psi_G \rangle \qquad (1)$$

for all such O_M. This condition has a clear-cut physical interpretation, but it involves matrix elements of a multitude of observables. One may ask whether it can be reformulated in terms of a single observable whose eigenvectors corresponding to a fixed eigenvalue, say 0, rep-

0031-9007/94/73(5)/613(4)$06.00

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resent all states which coincide with G in the given region. In other words, does there exist a positive operator, $O_{G \text{ inside } \mathcal{R}?}$, such that

$$\langle \psi_S | O_{G \text{ inside } \mathcal{R}?} | \psi_S \rangle = 0, \tag{2}$$

if and only if S looks like G inside $\mathcal{R}$? That such operators exist is taken for granted by Hegerfeldt; cf. his remarks about the projection operator O_{eB} in [1]. But the situation is not so simple. In fact, there are marked differences between nonrelativistic and relativistic theories which manifest themselves in the following alternative structure of the set of vectors ψ_S satisfying condition (1).

(i) Linear combinations of vectors ψ_S satisfying (1) again satisfy this condition (after normalization). This case is generic in nonrelativistic quantum field theory, where it holds for a total set of states G. These states describe a situation where one has locally maximal information about the underlying system (locally pure states). Examples are the Fock vacuum and all coherent states. For any such state G one can introduce a corresponding operator $O_{G \text{ inside } \mathcal{R}?} = 1 - \sum_{n=1}^{\infty} |\psi_S^n\rangle\langle\psi_S^n|$, where ψ_S^n, $n = 1, 2, \ldots$, is some orthonormal basis in the subspace of the physical Hilbert space spanned by the vectors ψ_S satisfying condition (1). This projection operator can be used to decide whether some arbitrary state A coincides with G inside of $\mathcal{R}$: one simply has to calculate the transition probability $\langle \psi_A | O_{G \text{ inside } \mathcal{R}?} | \psi_A \rangle$ and to check whether it is equal to 0. A positive value would indicate a local deviation of A from G. Thus in the nonrelativistic setting there exist operators which completely fix the local properties of states and it is then possible to study these properties in terms of transition probabilities.

(ii) Certain normalized linear combinations of vectors ψ_S satisfying condition (1) do not comply with this condition. This is the case in RQFT for *every* choice of G [4]. It is an important consequence of this fact that all states look locally like mixtures. For let ψ_S and $\psi_{S'}$ be vectors satisfying (1) such that $c\psi_S + c'\psi_{S'}$ no longer complies with this condition for certain complex numbers c, c'. As is easily verified there holds

$$\langle \psi_+ | O_M | \psi_+ \rangle + \langle \psi_- | O_M | \psi_- \rangle = \langle \psi_G | O_M | \psi_G \rangle, \tag{3}$$

where $\psi_\pm = (2|c|^2 + 2|c'|^2)^{-1/2}(c\psi_S \pm c'\psi_{S'})$. Hence the expectation values of the observables O_M in the state G can be interpreted in terms of a mixture of states which differ locally from G. As a matter of fact, every state G is locally a mixture of an infinite number of such states. In contrast to the nonrelativistic case, therefore, it is not possible to fix the local properties of states with the help of (projection) operators. The assumption that the states satisfying condition (1) correspond to vectors which are annihilated by some operator is intrinsically inconsistent in RQFT.

In view of these facts one is forced to base the local analysis of states, which is fundamental in any discussion of causal effects, on a comparison of states in the sense of relation (1). The nonvanishing of expectation values of positive operators (transition probabilities), as considered by Hegerfeldt, is not an adequate criterion to study this issue in relativistic theories [5]. This important point may be illustrated by a simple example. If G is, e.g., the vacuum state in RQFT, then the expectation value $\langle \psi_G | O_M | \psi_G \rangle$ cannot vanish for *any* positive operator O_M corresponding to a localized measurement [6]. To test for a local deviation of S from G one can therefore *not* take as a criterion that the expectation value of some suitable projection operator (or, more generally, some positive operator) has a nonzero expectation value in the state S. For this expectation value would be nonzero even if the vacuum G is present. This point has been overlooked by Hegerfeldt and led him to deduce from Eq. (8) in [1] an apparent causality paradox. A deviation of S from G would show up, however, in different values of the left- and right-hand sides of (1) for some O_M [7].

After these general remarks let us turn now to the actual discussion of Fermi's gedanken experiment. Let X be the ground state of an isolated atom which is localized in the vicinity of 0 and surrounded by vacuum and let ψ_X be the corresponding state vector. Following Fermi, we consider a state S, described by a vector ψ_S, which looks inside a ball $\mathcal{R}$ of radius R about 0 like X; i.e.,

$$\langle \psi_S | O_M | \psi_S \rangle = \langle \psi_X | O_M | \psi_X \rangle \tag{4}$$

for all observables O_M which are localized in $\mathcal{R}$. In the complement $\mathcal{R}^c$ of this ball S may look like any other state Y, e.g., like some excited atom. If, as expected, the subsystem in $\mathcal{R}^c$ does not affect the atom in $\mathcal{R}$ within the time interval $0 < t < R/c$ it should not be possible to discriminate S from X by any measurement M' which one preforms at time t within the ball $\mathcal{R}_t$ of radius $R - ct$ about 0. Phrased differently, S should still look like an atom in its ground state within the smaller region $\mathcal{R}_t$. Hence, using the Heisenberg picture, it should hold in the theoretical setting that

$$\langle \psi_S | O_{M'}(t) | \psi_S \rangle = \langle \psi_X | O_{M'}(t) | \psi_X \rangle, \tag{5}$$

where

$$O_{M'}(t) = e^{itH/\hbar} O_{M'} e^{-itH/\hbar}. \tag{6}$$

It is a fundamental fact that relation (5) is a consequence of relation (4) in theories where the underlying field equations are hyperbolic. Within the setting of RQFT this fact is called "primitive causality" [8] and has been established in models; cf., for example, [9]. It is independent of the spectral properties of the generator H, which in fact depend on the systems which one considers (few body systems, thermal states, etc.). Hence in this

respect the predictions of RQFT are in perfect agreement with the ideas of Fermi.

There remains, however, the question of whether the theory is capable of describing the physical situation envisaged by Fermi; cf. point (c) in [1]. Given two vectors ψ_X, ψ_Y corresponding to states X, Y, does there not exist a vector ψ_S describing the *composite* state S which looks like X in a given region $\mathcal{R}$ and like Y in its complement $\mathcal{R}^c$? These requirements fix ψ_S completely and can be cast into the following condition on the expectation values,

$$\langle \psi_S | O_M \times O_{M^c} | \psi_S \rangle = \langle \psi_X | O_M | \psi_X \rangle \langle \psi_Y | O_{M^c} | \psi_Y \rangle, \quad (7)$$

where O_M and O_{M^c}, respectively, denote operators corresponding to measurements in $\mathcal{R}$ and $\mathcal{R}^c$. Relation (7) gives formal expression to the idea that S is composed of states X and Y which are localized [in the sense of condition (1)] in disjoint regions and do not "overlap"; cf. point (a) in [1].

The question of whether such product states exist is known in RQFT as the problem of "causal (statistical) independence" [10]. It has an affirmative answer [11], but the vectors ψ_S have in general infinite energy even if ψ_X and ψ_Y have finite energy. This phenomenon can be traced back to the uncertainty principle and may be easily understood in the framework of nonrelativistic quantum mechanics: if ψ_X and ψ_Y are the configuration space wave functions of distinguishable systems then the wave function ψ_S of the composite state is given by

$$\psi_S(\mathbf{x}, \mathbf{y}) = N \times \begin{cases} \psi_X(\mathbf{x})\,\psi_Y(\mathbf{y}), & \text{for } \mathbf{x} \in \mathcal{R}, \mathbf{y} \in \mathcal{R}^c, \\ 0, & \text{otherwise}, \end{cases} \quad (8)$$

where N is a normalization constant. This function has in general a discontinuity when $\mathbf{x}$ or $\mathbf{y}$ are at the boundary of $\mathcal{R}$, unless the wave functions ψ_X and ψ_Y both happen to vanish at these points. (Note that wave functions of states with sharp energy, such as bound states, in general do not have such nodes.) As a consequence, the expectation value of the Hamiltonian becomes infinite. In RQFT the situation is even worse because of pair creation. There it turns out that due to such processes the vector ψ_S cannot be an element of the physical Hilbert space $\mathcal{H}$ describing few body systems (e.g., Fock space in free field theory).

Thus the theory predicts that every member of the ensemble described by S has infinite energy. Hence a preparation of this state would not be possible in practice. This fact seems to be in conflict with the ideas of Fermi, but the apparent difficulty disappears if one notices that for the determination of c it suffices to consider "tame" states T which look like X in a region $\mathcal{R}_<$ and like Y in $\mathcal{R}_>^c$, where $\mathcal{R}_<$ is slightly smaller an $\mathcal{R}_>$ slightly larger than $\mathcal{R}$. It is not really necessary to completely fix the state T in the layer between these two regions. By making this layer sufficiently small one can then determine c, as outlined above, with arbitrary precision.

It has been shown in RQFT under very general conditions that there exist vectors ψ_T in the physical Hilbert space $\mathcal{H}$ which satisfy condition (7) for the slightly smaller regions [12]. The existence of such vectors has also been established in models [13]. Thus also in this respect Fermi's gedanken experiment poses no theoretical problems.

It should be mentioned that the states T have to be carefully adjusted in the layer between the two regions $\mathcal{R}_<$ and $\mathcal{R}_>^c$ to smoothly interpolate between the states X and Y and thereby be representable by vectors ψ_T in the Hilbert space $\mathcal{H}$. This adjustment may be viewed as the process of "renormalization," indicated in point (b) of [1], which surrounds state X by some "cloud." In more physical terms, any state of the type considered by Fermi which can actually be prepared in an experiment necessarily contains, besides the two atoms, other particles, e.g., photons. This inevitable creation of particles in the process of localizing physical systems is the basic reason for the absence of locally pure states and the ensuing uselessness of the concept of transition probability for the study of local properties. But it is not at variance with the existence of a maximal propagation speed c.

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[3] We assume that G exists as a global state described by a vector in the physical Hilbert space. This assumption causes no problems if one considers configurations of atoms. But it would require further discussions if one thinks, e.g., of quarks as subsystems of hadrons (confinement).
[4] These differences between relativistic and nonrelativistic theories are encoded in the specific structure of the algebras generated by the respective observables in bounded regions. In nonrelativistic quantum field theory these algebras are generically of *type I* according to the classification scheme of von Neumann. They are distinguished by the existence of minimal projections which do not dominate any other nonzero projection in the algebra. In RQFT the algebras corresponding to bounded regions are of *type III*; i.e., every projection operator in the algebra dominates a strictly smaller one of the same relative dimension. Phrased differently, all projections in the algebra have infinite relative dimension. Cf., for example, Sec. III. 2 and Sec. V. 6 in R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992).
[5] This remark applies also to nonrelativistic theories if one wants to consider locally mixed states.
[6] This is the statement of the Reeh-Schlieder theorem; cf. the preceding reference. As a matter of fact the statement holds for almost any physical state G (the set of such states is of *second category*).
[7] A quantitative measure for the observed differences between S and the given state G may be defined as fol-

lows: if $\mathcal{A}$ denotes the algebra generated by the respective observables one puts $d_{\mathcal{A}}(S,G) = \sup|\langle\psi_S|O_M|\psi_S\rangle - \langle\psi_G|O_M|\psi_G\rangle|$, where the supremum is to be taken with respect to all operators $O_M \in \mathcal{A}$ whose operator norm is less than or equal to 1. This quantity defines a distance between the states S and G which can assume values between 0 and 2. A closely related quantity, which may be called quasitransition probability, is the expression $P_{\mathcal{A}}(S,G) = 1 - \frac{1}{4}d_{\mathcal{A}}(S,G)^2$. It is a certain substitute for the familiar transition probability of states in those cases, where the algebra of observables $\mathcal{A}$ does not contain minimal projections or the states are not pure. In the special case where $\mathcal{A}$ is the algebra of all bounded operators on the underlying Hilbert space the quasitransition probability coincides with the familiar quantum mechanical transition probability $|\langle\psi_S|\psi_G\rangle|^2$. But in general it cannot be expressed in terms of expectation values of some operator O_G in the states S; cf. J. E. Roberts and G. Roepstorff, Commun. Math. Phys. **11**, 321 (1969).

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