Quantum Spherical Models for Dirty Phase Transitions

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We construct large-n (spherical) limits for a series of interesting quantum phase transitions in disordered systems, including quantum ferromagnets and spin glasses, superconducting thin films with and without an external magnetic field, the dirty boson problem, the fractional quantum Hall effect, and Nelson's model of flux lines in high-temperature superconductors. The spherical limit always produces a random matrix inversion problem with self-consistency conditions, which then must be solved numerically. We present preliminary results for the dirty boson problem in two spatial dimensions.

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One of the few remaining analytic tools that has not been applied to phase transitions in dirty quantum systems is the large-n expansion. This technique has been applied, with great success, to a large number of *clean* two-dimensional quantum spin systems [1] and *infinite range* quantum spin glass models [2, 3]. In what follows we describe a series of models which are large n, or spherical limits, of particular classes of *short-range* quantum models with quenched disorder. Here n is the number of order-parameter components, and the limit $n \to \infty$ is taken in the interest of obtaining a tractable model which still contains much of the original physics. As described below, an astonishing variety of interesting problems may be attacked by this method: superfluidity of ^{4}He in porous media (the dirty boson problem), flux phases in high-temperature superconductors (especially the proposed vortex glass phase), universal transport phenomena in superconducting films (both with and without external magnetic fields), and quantum spin glasses and antiferromagnets. It turns out that solving these models requires the inversion of very large matrices with near-diagonal disorder, along with the imposition of a self-consistency constraint. Such inversions can be done numerically, and the self-consistency condition imposed via an iteration procedure. The size of the matrix grows as a power of the system size, and standard finite-size scaling techniques must be used to extract the critical behavior. The limit $n \to \infty$ in certain problems is known to produce pathologies, but we find that wherever general results are known, indications are that the corresponding spherical limit exhibits all the required properties, sometimes in novel and surprising ways. In this Letter we construct the large-n limits for all of these models and then present preliminary data for the dirty boson problem.

(1) *Classical spins.*—The usual reduced exchange Hamiltonian for a lattice of classical spins, $\mathbf{s}_i$, is $\bar{\mathcal{H}}_{\rm cl} \equiv \beta \mathcal{H}_{\rm cl} = -\frac{1}{2}\sum_{i,j} K_{ij}\mathbf{s}_i \cdot \mathbf{s}_j$, where $\beta = 1/k_BT$, $K_{ij} = \beta J_{ij}$ is the matrix of (reduced) exchange interactions, and the spins are n-component vectors whose length we take to be $|\mathbf{s}_i| = \sqrt{n}$. Of interest here is the case when the interactions are short ranged and have a *random* aspect.

In order to study the large-n behavior it is convenient to write the partition function in the form $Z = \mathrm{tr}^{\mathbf{s}}[e^{-\bar{\mathcal{H}}}] = \prod_i \int d^n s_i\, \delta(|\mathbf{s}_i|^2 - n)\, e^{-\bar{\mathcal{H}}_{\rm cl}[\mathbf{s}]}$. Using the representation $\delta(x) = \int_{-\infty}^{\infty}(d\lambda/2\pi)e^{i\lambda x}$ we may then factor the trace over $\mathbf{s}$ into n independent traces over each component of $\mathbf{s}$, at the expense of introducing a new trace over a one-component field λ: $Z = \mathrm{tr}^{\lambda}[e^{-n\mathcal{F}[i\lambda]}]$; $\mathrm{tr}^{\lambda} \equiv \prod_i \int d\lambda_i/2\pi$. Here

$$\mathcal{F}[i\lambda] = i\sum_i \lambda_i + \tfrac{1}{2}\ln\det(\kappa/2\pi); \qquad (1)$$

s is any single component of $\mathbf{s}$, and $\kappa_{ij} = -2i\lambda_i\delta_{ij} - K_{ij}$. For large n the method of steepest descents is appropriate, and we seek the saddle point of $\mathcal{F}[i\lambda]$. With $\sigma_i = -2i\lambda_i$, the saddle point is defined by $0 = 2\,\partial\mathcal{F}/\partial\sigma_i = \langle s_i^2\rangle - 1$, implying $(\kappa^{-1})_{ii} = 1$ for each i [4]. If there are N spins, this represents N equations for the N σ_i (which clearly will be real). For noninfinite n, corrections in powers of $1/n$ may in principle be computed by considering fluctuations of the λ_i about the saddle point values $i\sigma_i/2$; however, in the random case even the $n \to \infty$ limit is not analytically tractable, and obtaining higher order corrections is even more difficult. In this Letter we will be concerned only with the spherical limit.

The free energy (1) is that of a constrained Gaussian model. The phase transition, at inverse temperature $\beta = \beta_c$, takes place when the eigenstate associated with the smallest eigenvalue, $\epsilon^0(\beta) = 0$, of the matrix κ first becomes extended. One may define various critical exponents to describe the form of the critical low-energy density of states, $\rho(\epsilon;\beta_c) \sim \epsilon^y$, the divergence of the localization length, $\xi \sim (\beta_c - \beta)^{-\nu}$, and so on. In the ordered state, which occurs for $\beta > \beta_c$, the system "condenses" into the eigenvector associated with this vanishing eigenvalue. If $J_{ij} > 0$, the ordered state is ferromagnetic in character.

Although there are some interesting classical problems that deserve future attention, the really interesting (and

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less computationally intensive) problems come from generalizing the formalism to quantum mechanical systems at $T = 0$.

(2) *The random-rod problem, or particle-hole symmetric dirty bosons.*—The first quantum mechanical model we will consider is most simply defined by $\bar{\mathcal{H}}_{\text{cl}}$ in which the nearest neighbor couplings are random, but *constant* along a particular direction, $\hat{\tau}$. This model is equivalent to those with columnar disorder discussed in [5]. To make more direct contact with the physics of bosons we consider instead a closely related version of this model defined by the reduced Hamiltonian

$$\bar{\mathcal{H}}_R = \int_0^\beta d\tau \Big[K \sum_i |\partial_\tau \boldsymbol{\psi}_i|^2 - \sum_{i,j} J_{ij} \boldsymbol{\psi}_i^* \cdot \boldsymbol{\psi}_j \Big], \tag{2}$$

where $\boldsymbol{\psi}_i(\tau)$ is an m-component *complex* vector (so that $n = 2m$) with $|\boldsymbol{\psi}_i|^2 = m$, $0 \le \tau < \beta$ is a continuous imaginary-time variable, and the index i runs over a d-dimensional lattice.

The large-n limit is obtained using the same representation for the delta function. This introduces a new field $\lambda_i(\tau)$, and at the saddle point we will have $\lambda_i(\tau) = i\sigma_i$, independent of τ, satisfying $1 = \beta^{-1} \sum_n [\kappa^{-1}(\omega_n)]_{ii}$, where $\kappa_{ij}(\omega_n) = (K\omega_n^2 + \sigma_i)\delta_{ij} - J_{ij}$, and $\omega_n = 2\pi n/\beta$ are the Matsubara frequencies. If we define the matrix $D_{ij} = \sigma_i \delta_{ij} - J_{ij}$, and its spectrum, $\sum_j D_{ij} \phi_j^\alpha = \epsilon^\alpha \phi_i^\alpha$, where α labels the eigenstates, then the ω_n sum may be performed exactly [6] to yield

$$\begin{aligned} 1 &= \sum_\alpha |\phi_i^\alpha|^2 \coth\!\Big(\tfrac{1}{2}\beta\sqrt{\epsilon^\alpha/K}\Big)/2\sqrt{K\epsilon^\alpha} \quad (\text{for each } i) \\ &= \sum_\alpha |\phi_i^\alpha|^2 / 2\sqrt{K\epsilon^\alpha}, \quad \beta \to \infty \; (T \to 0). \end{aligned} \tag{3}$$

The eigenvalues $\epsilon^\alpha(K; \{\sigma_i\})$ are the effective single particle energies (renormalized by the repulsive interactions), and ϕ^α are the corresponding eigenstates. As K increases the ϵ^α decrease, and the critical point, $K = K_c$, again occurs when the localization length of the zero eigenvalue diverges.

Very little is known about the random-rod model in dimensions $d \ge 2$. Analytic results have been obtained only from an extremely poorly behaved double-epsilon expansion [5]. For $d = 2$ the model is relevant to particle-hole symmetric Josephson junction array models, and, more importantly, to certain types of disordered two-dimensional quantum antiferromagnets [1] and spin glasses [2].

(3) *Dirty bosons.*—The generalization of (2) appropriate to bosons is the complex action (closely related to a functional integral representation of the second-quantized boson Hamiltonian [7]),

$$\begin{aligned} S_B = -\int_0^\beta d\tau \Big[K \sum_i \boldsymbol{\psi}_i^* \cdot (\partial_\tau - \mu_i)^2 \\ \times \boldsymbol{\psi}_i + \sum_{i,j} J_{ij} \boldsymbol{\psi}_i^* \cdot \boldsymbol{\psi}_j \Big], \end{aligned} \tag{4}$$

where, again, $|\boldsymbol{\psi}_i|^2 = m$, $\mu_i = \mu - w_i$, μ being the chemical potential, and w_i the (random) external site potential. The main effect of the μ_i is to introduce linear time derivative terms, $\mu_i \boldsymbol{\psi}_i^* \cdot \partial_\tau \boldsymbol{\psi}_i$, which break particle-hole symmetry, and make the action complex. The coupling $K = 1/2u_0$ is inversely proportional to the repulsive on-site potential, u_0, in the quantum boson Hamiltonian.

The large-n limit is obtained precisely as before, but now with matrix $\kappa_{ij} = [\sigma_i - K(i\omega_n - \mu_i)^2]\delta_{ij} - J_{ij}$. If the μ_i are random, the ω_n sum cannot be performed exactly: A different matrix must be diagonalized for each ω_n. A *statistically* particle-hole symmetric model results when $\mu = 0$, and the w_i have an even probability distribution, although models with $\mu = 0$ and $\mu \ne 0$ are expected to lie in the same universality class [8]. Numerically, however, a much more convenient model results when $\mu_i \equiv \mu \ne 0$ is uniform. So long as J_{ij} remains random, this model too will lie in the same universality class. The advantage now is that, using the same matrix D_{ij} and its spectrum, the ω_n sum may now be performed exactly to yield

$$\begin{aligned} 1 &= \sum_\alpha \frac{|\phi_i^\alpha|^2}{4\sqrt{\epsilon^\alpha K}} \Big\{ \coth\Big[\tfrac{1}{2}\beta\Big(\sqrt{\epsilon^\alpha/K} + \mu\Big)\Big] \\ &\qquad + \coth\Big[\tfrac{1}{2}\beta\Big(\sqrt{\epsilon^\alpha/K} - \mu\Big)\Big]\Big\} \\ &= \sum_\alpha \frac{|\phi_i^\alpha|^2}{2\sqrt{K\epsilon^\alpha}} \Theta\Big(\sqrt{\epsilon^\alpha/K} - \mu\Big), \qquad \beta \to \infty \; (T \to 0), \end{aligned} \tag{5}$$

where $\Theta(x) = 1$ for $x \ge 0$, vanishing otherwise, is the step function. We now argue that, in fact, this model has *two* phase transitions at $T = 0$. When K is sufficiently small, $K < K_M$, all ϵ^α lie above $|\mu|$, and the $T = 0$ forms of (5) and (3) are *identical*: the thermodynamics is completely independent of μ, and the system is therefore *incompressible*. This latter property defines the famous *Mott insulating* phase [7] with fixed integer density (which we take to be exactly zero). After ϵ^0 hits $K\mu^2$ at $K = K_M$ it *sticks* at that value, and as K increases further the eigenvalues above it experience level repulsion. The system is now in the *Bose glass* phase [7]. One can show that this level repulsion leads to a *positive* density of states, $\rho(\epsilon)$, at $\epsilon = K\mu^2$, as predicted by the general theory [7]. Although there is now a "condensate fraction" in the state ϵ^0 [and (5) must be modified appropriately], which is proportional to the total *density*, superfluidity does not occur until $K = K_c > K_M$ where this state finally produces long-range order. Below we present numerical results for this model.

(4) *Magnetic field tuned transitions in thin-film superconductors.*—An interesting generalization of the dirty boson problem arises when one considers a dirty superconducting film in a perpendicular magnetic field $\mathbf{B}$ [9].

Since the bosons, in our picture, are charge $2e$ Cooper pairs, they couple to the magnetic field via the usual $[\mathbf{p} - (2e/c)\mathbf{A}]$ generalization of the momentum. Here $\mathbf{A}$ is the vector potential, with $\mathbf{B} = \nabla \times \mathbf{A}$ assumed to be uniform. When space is discretized, the hopping matrix acquires a phase factor, $J_{ij} \rightarrow J_{ij} e^{i2eA_{ij}/c}$, where $A_{ij} = -A_{ji}$ is the vector potential on the bond joining sites i and j. The diagonalization procedure leading to Eq. (3) is now valid, with $D_{ij} = \sigma_i \delta_{ij} - J_{ij} e^{i2eA_{ij}/c}$.

Unlike the $\mathbf{B} = 0$ problem, the spherical limit offers a more or less *unique* method for studying the phase transition: The complex hopping matrix seems to give rise to sign problems in Monte Carlo simulations just as intractable as those for fermions. Other than purely phenomenological information provided by various duality arguments [9], nothing is known theoretically about the transition.

(5) *The fractional quantum Hall effect.*—The fractional quantum Hall effect has a deep connection with superfluidity of particles with fractional statistics (anyons) in a strong magnetic field [10]. Thus, via the usual singular gauge transformation, one may represent the anyons as bosons with magnetic flux tubes attached. The boson action then acquires a statistical gauge field, $\mathbf{a} \equiv (a_0, a_x, a_y)$, whose dynamics is governed by the famous Chern-Simons term in the action [10]. Generalizing the derivation of (4) to this case, one obtains the same form, but with $\partial_\tau - \mu_i \rightarrow \partial_\tau - \mu_i - ia_{0,i}$, and $J_{ij} \rightarrow J_{ij} e^{i(2eA_{ij}/c + a_{ij})}$. For general m the appropriate Chern-Simons term is then $(im/4\theta)\epsilon^{\alpha\beta\gamma} a_\alpha \partial_\beta a_\gamma$ (appropriately discretized), where θ is the statistical parameter, and the Greek indices run over both space and time. One may now investigate the transition between the fractional quantum Hall state and the insulating state. In the absence of disorder, and $\mu_i \equiv 0$, the quantum Hall-Mott insulator transition was investigated in [11]. It was found that, at order $1/m$, universal quantities acquire a θ dependence, violating basic assumptions contained in earlier work. An open question is whether the same is true in the presence of disorder. Since we are able to treat only the limit $m \rightarrow \infty$, we cannot really address this question for this model. However, a second generalization of the model, in which there are m statistical gauge fields, one for each component of $\boldsymbol{\psi}$, was also considered in [11]. This model breaks the SU(m) symmetry, and so may not have a proper $1/m$ expansion, but it does have a well-defined $m \rightarrow \infty$ limit. This limit is soluble only perturbatively for small θ ($\theta = 0$ corresponding to the original boson problem) and the Hall conductance was found to have a leading linear dependence on θ with a universal coefficient [11]. It is straightforward to write down an expression for the coefficient of this linear term in the presence of disorder as well. It is given by a multiple sum over products of four dirty boson Green's functions, and there is no apparent reason why it should vanish. We are in the process now of computing it numerically.

(6) *Flux phases in high-temperature superconductors, or bosons with time varying disorder.*—This final application is, numerically, the most difficult to address, but perhaps the most physically interesting. In the fully screened limit in which the flux lines are far apart (separation much larger than the magnetic penetration depth Λ) one may write down an effective Hamiltonian (rather different from those of the unscreened "gauge glass models" studied by others) for N flux lines in a type-II superconductor [12]:

$$\bar{\mathcal{H}}_F = \int_0^L \left[\frac{m}{2} \sum_{i=1}^N |\partial_z \mathbf{r}_i(z)|^2 + \frac{1}{2} \sum_{i \neq j} v[\mathbf{r}_i(z) - \mathbf{r}_j(z)] + \sum_{i=1}^N w[\mathbf{r}_i(z), z] \right] dz , \tag{6}$$

where $\mathbf{r}_i(z)$ is the position of the ith flux line as a function of height, z, which also defines the direction of the magnetic field, $\mathbf{H}$; $v(\mathbf{r})$ is the effective interaction between flux lines, and $w(\mathbf{r}, z)$ is the external (e.g., random impurity) potential. The phenomenological tilt modulus m tends to align the flux lines along the field, and their number is set by the magnitude of $\mathbf{H}$. If periodic boundary conditions in z are imposed, or simply if $L \rightarrow \infty$, $\bar{\mathcal{H}}_F$ is precisely the Feynman path integral representation of the imaginary-time-dependent quantum boson Hamiltonian,

$$\hat{\mathcal{H}}_B(\tau) = \sum_{i=1}^N \frac{|\hat{\mathbf{p}}_i|^2}{2m} + \frac{1}{2} \sum_{i \neq j} v(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j) + \sum_{i=1}^N w(\hat{\mathbf{x}}_i, \tau) ,$$

where z is now identified with τ, and the width L is identified with the inverse temperature β. The usual second-quantized form then follows immediately, as do all the standard functional integral representations based on it. In particular, (4) still holds, but now with τ-dependent site potentials $\mu_i(\tau) = \mu - w_i(\tau)$, and $\mu \propto H$. The saddle point values $\sigma_i(\tau)$ are now τ dependent, and the full matrix $\kappa_{ij}(\tau, \tau') = (\{\sigma_i(\tau) - K[\partial_\tau - w_i(\tau)^2]\} \times \delta_{ij} - J_{ij})\delta(\tau - \tau')$ (appropriately discretized) must be inverted in order to impose the self-consistency condition, $(\kappa^{-1})_{ii}(\tau, \tau) = 1$.

Perhaps the most interesting open question is whether or not the model actually *has* a phase transition when $L \rightarrow \infty$ (i.e., $T \rightarrow 0$ in the analogous boson problem). Thus, whether there exist an *insulating* phase (which would correspond to the superconducting vortex glass phase) [13], intermediate between the Meissner and normal phases, is not clear for $d \geq 2$. Although columnar disorder is known to localize the boson world lines [14], time-dependent disorder may serve actually to *delocalize* them. In $d = 1$ there *is* a vortex glass phase [13], but, contrary to superfluid phases, localized phases tend to be *less* stable in higher dimensions [15]. If the model *does* have a transition there are also a number of interesting questions to be addressed. For example, are the correlations isotropic, $\nu_\tau = \nu$? In all existing scaling analyses of experimental data this equality is assumed.

We end by demonstrating our method on the two-dimensional dirty boson problem. In Fig. 1 we show data for the correlation length and superfluid density, obtained by averaging over at least 150 realizations of the disorder, for linear system sizes up to $L = 14$. Our method has the advantage that we may set $T = 0$ from the outset, allowing us to perform finite-size scaling only in L. By standard arguments [16], the crossover point of the ξ/L curves for different L represents the critical value of K. We find $K_c = 0.0925 \pm 0.008$. The superfluid density scales as $L^z \rho_s$, and we may now determine the dynamical exponent z by demanding that the scaled superfluid density curves cross at the same critical value, K_c. We find a result, $z = 2.0 \pm 0.1$, consistent with the prediction, $z = 2$ (used in the figure), of the general theory [7]. Scaling also predicts that ξ/L and $L^z \rho_s$ are universal functions of the variable $L^{1/\nu}(K - K_c)$, where ν is the correlation length exponent. We find $\nu = 1.0 \pm 0.1$, consistent with the predicted inequality $\nu \geq 1$ [17]. Using $z = 2$ we compute also one of the universal conductances [18], $\sigma^* = (4e^2/h) \lim_{\omega \to 0} \rho_s(\omega)/\omega$, where $\rho_s(\omega)$ is the frequency dependent superfluid density [16]. We find $\sigma^* \simeq 0.18$, which should be compared to the result $\sigma^* \simeq 0.14$ for $m = 1$ [16]. This is *not* the universal conductance inferred experimentally [9], which involves an extrapolation from finite temperature, but is the easiest to compute. We will present experimentally more relevant results in a future publication.

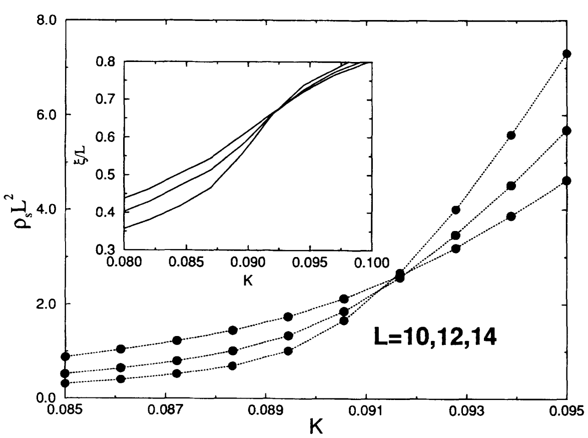

FIG. 1. Finite-size scaling analysis for the dirty boson problem. The inset shows the scaled correlation length for system sizes $L = 10$, 12, 14 (steeper curves correspond to larger system size). The crossover point determines the critical coupling, $K_c = 0.0925 \pm 0.008$ (the Mott transition occurs at approximately $K_M \simeq 0.085$). The main plot shows that the choice $z = 2$ gives a consistent crossover point for the superfluid density. Error bars are about the size of the symbols.

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