# Bifurcations and Spatial Chaos in an Open Flow Model

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It is shown that a coupled map model for open flow may exhibit spatial chaos and spatial quasiperiodicity with temporal periodicity. The locations of these patterns, which cover a substantial part of parameter space, are indicated in a comprehensive phase diagram. In order to analyze the encountered phenomena, a novel class of spatial maps is introduced which is very efficient in accurately reproducing the original spatial patterns. It is found that temporally period one spatial chaos is convectively unstable, and that it is possible to predict an essential aspect of the bifurcation behavior of the coupled system solely by considering its corresponding spatial map.

For the understanding of many complex systems in nature, the study of simple chaotic paradigms has proven to be an invaluable tool. Sometimes, substantial insight can be gained by only considering temporal aspects. In many cases, however, it is imperative to also explicitly take space into consideration.

In recent years this has led to great interest in the dynamics of coupled map lattices (CML) which are not only spatiotemporal, but can also be classified by a distinct number of universality classes. Particularly, coupled logistic maps have been studied widely for their combination of computational efficiency and phenomenological richness [1–4].

In this Letter, we will investigate the nature of spatial chaos by introducing a class of spatial maps which can accurately reproduce the spatial patterns of the one-way coupled logistic lattice (OCLL) in parameter regions where the latter is temporally periodic.

In the different context of a proposal for a distributed optical element system, Otsuka and Ikeda [5] also discussed the stability of spatial chaos and pointed out the importance of the relationship between spatial Lyapunov exponents and dynamic stability (for other work related to spatial chaos see also, e.g., [6]). Otsuka and Ikeda's conclusion that in their system spatial chaos is dynamically unstable in the case of unidirectional coupling is confirmed by our findings which indicate that spatial chaos is unstable in a range of moving frames.

The OCLL, due to its conceptual reminiscence with open fluid flow often referred to as a model for open flow, is defined as

$$x^i_{n+1} = (1 - \epsilon)f(x^i_n) + \epsilon f(x^{i-1}_n),\qquad (1)$$

where the local element is the logistic map $f(x_n) = 1 - \alpha x^2_n$. The parameters are the nonlinearity $\alpha$ and the coupling constant $\epsilon$, while $i$ is the index for the lattice sites, and $n$ the discrete time.

Many interesting features of (1), like spatial period doubling and selective amplification of noise, were already reported in [7] and [8–10], but those investigations did not take the full range of possible coupling constants ($0.0 \le \epsilon \le 1.0$) into account and were generally limited

to $\epsilon \lesssim 0.5$. Recent studies of the diffusively coupled logistic lattice, of which the OCLL is in principle nothing but a version with a maximally asymmetric coupling, however, have shown that larger values of $\epsilon$ may yield unexpected phenomena like the traveling wave or suppression of supertransient chaos at high nonlinearity [11,12].

In the OCLL, for larger values of $\epsilon$, we have found the fascinating novelty of spatial chaos with temporal periodicity (see also [13]). Consequently, with regard to its spatiotemporal properties, the dynamics of the OCLL for nonlinearities past the accumulation point of the single logistic map can be described as dominantly temporal ($0.0 \le \epsilon \le 0.05$), spatiotemporal ($0.05 \lesssim \epsilon \lesssim 0.45$), or spatial ($0.45 \lesssim \epsilon \le 1.0$).

A rough phase diagram for the OCLL is given in Fig. 1, where the basic types of patterns we would like to distinguish here are marked as follows: spatially chaotic and temporally periodic patterns (SC), spatially (and temporally) periodic patterns (SP), spatially quasiperiodic patterns with temporal periodicity (SQP), and spatiotemporal patterns (STP). The STP region includes various kinds of patterns like pattern selection with remnant chaos, spatiotemporal intermittency, and a large area of spatiotemporal chaos (STC). The region marked as "ZZ" is the well-known zig-zag pattern which is spatially and temporally period 2, and therefore a special case of
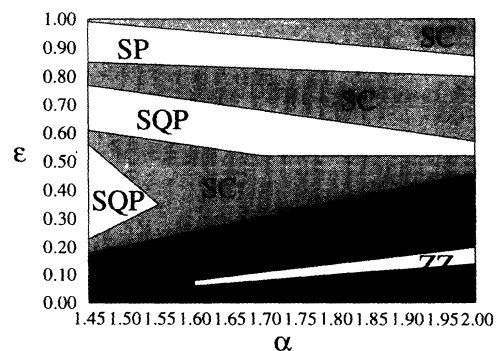


FIG. 1. Phase diagram for the open flow model. The system size is $N = 384$.

SP. Throughout, the left boundary was fixed to 0 (the exact value does not seem to matter much).

In order to illustrate the two temporally periodic and spatially nonperiodic patterns, space-amplitude plots of SC and SQP are displayed in Figs. 2 and 3, respectively. In the case of SC (of which a definition will be given below in terms of the newly introduced spatial map), the temporal periodicity rapidly bifurcates to 32 but then remains constant until the end of the lattice. In the case of SQP, after the initial bifurcations, the pattern has a temporal periodicity of 8 with a spatial period close to 14, while the points of the return map lie on an eye-shaped loop indicating quasiperiodicity.

The various areas in the phase diagram are separated by sharp lines in order to give some impression of their location. In fact, however, the lines are not very sharp at all and in many cases multiple attractors belonging to different basic patterns exist for the same parameters. In some cases this might be related to the dependence of the spatial locations of the temporal bifurcation points on the initial conditions, and thus mainly reflect the difference between upflow and downflow. In other cases, however, different types of patterns may exist for the same temporal periodicity, indicating genuine multiple states. The diagram furthermore only indicates the most common patterns in a larger area of parameter space. That is to say, small SQP regions within the SC region, e.g., were not taken into consideration.

In order to analyze the phenomenon of purely spatial chaos, we employ the one-way coupling and the temporal periodicity to introduce a spatial map as an implicit equation of equal-time spatial variables. If we define $F(x_n^i) = (1 - \epsilon)f(x_n^i) + \epsilon f(x_n^{i-1})$, then a lattice site $x_n^i$ which has a temporal periodicity $k$ must fulfill

$$x_n^i = F^k(x_n^i), \tag{2}$$

where $F^k(x_n^i)$ is the $k$th iterate of $F(x_n^i)$. We can now formally define a spatial map corresponding to Eq. (2) in an implicit form as

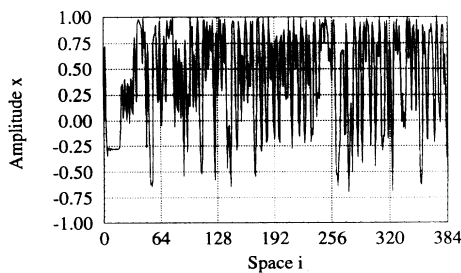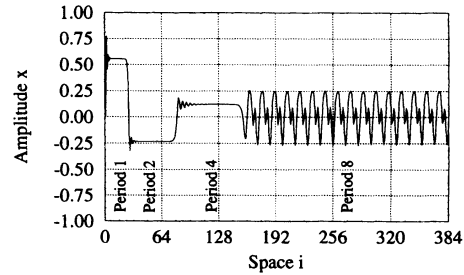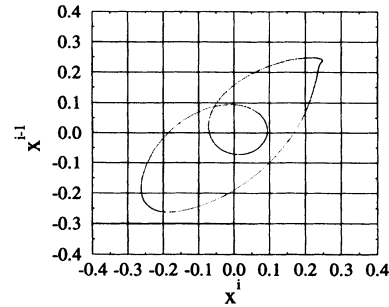$$G^k(x^i) = -x^i + F^k(x^i), \tag{3}$$

FIG. 2. Spatial chaos with temporal periodicity. $\alpha = 1.7$, $\epsilon = 0.45$, and $N = 384$. The temporal periodicities are 1 for $i = 1–2$, 2 for $i = 3–16$, 4 for $i = 17–31$, 8 for $i = 32–78$, 16 for $i = 79–127$, and 32 for $i = 128–384$, respectively.

(a)

(b)

FIG. 3. Spatial quasiperiodicity with temporal periodicity. (a) $\alpha = 1.45$, $\epsilon = 0.5$, and $N = 1024$. Only the first 384 sites are shown, the remaining sites are the same. The indicated periodicities are temporal periodicities. (b) Return map corresponding to (a) for sites $i = 256–1024$.

with which a lattice can be generated by successively finding the roots of Eq. (3) and incrementing the index $i$. Since Eq. (3) depends on $k$ variables, $x^{i-1}, \ldots, x^{i-k}$ need to be supplied as initial conditions (note the absence of the time index $n$). Thus we can reduce even an infinite lattice, and hence an infinite dimensional system, to a $k$-dimensional map, while substantially improving the opportunities for mathematical and numerical analysis. In general, of course, an equation like (2) will have a large number of solutions, especially if the periodicity $k$ is high. We will now show, however, that for sufficiently large $\epsilon$ there is only one.

The derivative of Eq. (3) is given by

$$G^{k\prime}(x^i) = -1 + (1 - \epsilon)^k(-2\alpha)^k \prod_{m=0}^{m=k-1} K^m(x^i), \tag{4}$$

where $K^m(x^i)$ is the $m$th iterate of $K(x^i) = (1 - \epsilon)f(x^i) + \epsilon f(x^{i-1})$ and $K^0(x^i) = x^i$. If $G^{k\prime}(x^i) < 0 \forall x^i$, there is at most one root. Since $|\prod_{m=0}^{m=k-1} K^m(x^i)| \leq 1$ this will be the case if $(1 - \epsilon)^k(-2\alpha)^k < 1$, i.e., if

$$\epsilon > 1 - \frac{1}{2\alpha}. \tag{5}$$

Thus, the spatial map (3) exactly represents the original model (1) as long as condition (5) holds [it should be noted that this condition covers a much larger area than

the region which has a temporal periodicity of one in the open flow model, and that it also holds for $k \to \infty$ in (3)].

Numerically, it was furthermore verified that for basically the entire temporally periodic region of the OCLL, the corresponding spatial maps had only one root [13], while the STP area yielded a great multiplicity of roots. We therefore believe that the arise of spatiotemporal patterns below $\epsilon_{\min}(\alpha)$ may be associated with the occurrence of multiple roots in the spatial map which prevent the system from reaching a periodic state.

With the help of our spatial map, spatial chaos can formally be defined by the requirement that $\lambda_G > 0$, where $\lambda_G$ is the spatial Lyapunov exponent of Eq. (3) (the temporal Lyapunov exponent for a temporally periodic lattice site will of course be negative, see also below).

In most cases, an analytical solution for Eq. (3) cannot be found. For $k = 1$, however, it is a quadratic equation, and the root is given by (the other root is outside of the allowed domain)

$$x^i = \frac{-1 + \sqrt{1 + 4(1 - \epsilon)\alpha[1 - \epsilon\alpha(x^{i-1})^2]}}{2(1 - \epsilon)\alpha}. \quad (6)$$

This forms a spatial map, which, contrary to Eq. (3), is of an explicit nature and therefore extremely efficient in generating the associated spatial patterns.

For values of the coupling constant $\epsilon$ where the OCLL in principle has a temporal periodicity of one, and where thus the spatial map (6) is a valid representation of the OCLL, Eq. (6) undergoes a bifurcation cascade to chaos as is shown in Fig. 4. These bifurcations are strictly in the spatial direction, and as such do not affect the temporal periodicity of the original model (1), and accordingly the possibility of spatial chaos is a natural consequence. It should be noted here that the spatial bifurcations mentioned above are fundamentally different from the spatial period doubling reported in [7]. In the former case, the temporal periodicity does not change, while a parameter does, and in the latter case, the period doubling occurs in the spatial direction and coincides with a doubling of the temporal periodicity, while the parameters remain the same.
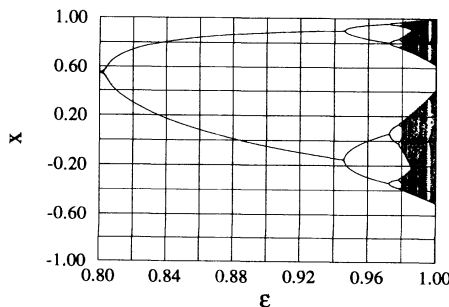


FIG. 4.   Bifurcation cascade of the $k = 1$ spatial map (6). The nonlinearity is $\alpha = 1.5$.

With the help of our spatial map, we can accurately reproduce the temporally periodic spatial patterns of the OCLL. In the OCLL, however, the temporal periodicity is not necessarily a constant throughout the lattice. If we consider Fig. 3, e.g., there are three homogeneous sections with temporal periodicities of 1, 2, and 4, respectively, and the spatially quasiperiodic pattern has a temporal periodicity of 8. Indeed, the $k = 1$ spatial map yields exactly the first section, the $k = 2$ the second, etc.

From this two important questions arise: Why does the OCLL bifurcate temporally, despite the existence of a stable and accurate solution in the spatial map, and, is it possible to use the data from the spatial map to predict whether the OCLL will undergo temporal bifurcations or not?

These questions are naturally related to the one of stability, and we will now show that both can be answered by considering comoving Lyapunov exponents [8] with regard to the spatial map. Conventionally, the spectrum of Lyapunov exponents would be determined by taking the logarithm of the eigenvalues of the product of Jacobi matrices, which, in this case would yield

$$\lambda^i = \log(1 - \epsilon) + \frac{1}{T} \sum_{n=1}^{n=T} \log f'(x_n^i), \quad (7)$$

where $T \to \infty$. For large $\epsilon$, the $\log(1 - \epsilon)$ will cause all the Lyapunov exponents to be negative and there seems to be no reason for the lattice to temporally bifurcate at some point downflow. A different approach is therefore necessary and the distinction between absolute and convective stability, which were shown to be essential concepts for the understanding of open flow [8] and which have no direct counterpart in the spatial maps, needs to be taken into account. A system is called absolutely unstable if it is unstable in any frame, and convectively unstable if it is only unstable in some moving frame.

Let us consider the $k = 1$ case. In a similar way as for the homogeneous fixed point state [2], it can be argued that, for large $\epsilon$, the Lyapunov exponent in the comoving frame that yields the maximum growth of an initial perturbation in the OCLL may be obtained as [13]

$$L_{\max} = \frac{1}{n} \sum_{i=0}^{i=n-1} \log|f'(x^i)|. \quad (8)$$

We can then generate a spatial pattern with the spatial map and use Eq. (8) to determine the pattern's stability in the comoving frame. Our numerical simulations indicate that if $L_{\max} < 0$, the pattern is absolutely stable, and a valid solution of the OCLL for all system sizes. If, however, $L_{\max} > 0$ (while $\lambda^i < 0$) the pattern generated by the spatial map is convectively unstable, and a valid solution of the OCLL only for those lattice sites which have the same temporal periodicity.

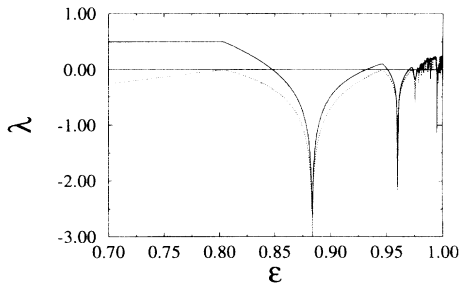In Fig. 5, the Lyapunov exponent according to Eq. (8) is plotted versus the coupling constant $\epsilon$ together with

FIG. 5. Lyapunov exponent versus the coupling constant $\epsilon$. The nonlinearity is $\alpha = 1.5$. Solid line: comoving frame. Dotted line: regular Lyapunov exponent of Eq. (6).

the regular exponent for Eq. (6). In both cases, the necessary values of $x^i$ were obtained with the help of the spatial map (6). Positive comoving Lyapunov exponents can clearly be seen for parameter values where the $k = 1$ spatial map is periodic, hence implying regions of convective instability. Our numerical results indicate that this corresponds to states of the OCLL which will temporally bifurcate. In this way, the spatial map provides us with an excellent tool to predict whether the original one-way coupled logistic lattice will bifurcate or not.

Since the Lyapunov exponent in the comoving frame is larger than the spatial one, it is implied that for the temporal period one case true spatial chaos does not exist. However, the dips between the bifurcation points, corresponding to superstable orbits which also exist in the periodic windows, extend to minus infinity, guaranteeing ample stability in their neighborhoods to conclude that spatial patterns arbitrarily long periods (i.e., in a practical sense virtually indistinguishable from chaos) can stably exist in the one-way coupled logistic lattice.

So far all numerical data obtained for our open flow model within the spatial chaos regime (including $k > 1$ cases) support the following conjecture.

*Conjecture.*—Spatial chaos is convectively unstable and subjected to further bifurcation downflow. A plausible argument for this conjecture is the following: If spatial chaos were stable, any sufficiently small perturbation of the state would decay in average over time at all lattice points. On the other hand, with the help of the spatial map, we can find a spatial sequence arbitrarily close to the original spatial pattern, which is also a periodic solution (attractor) of the original open flow model. Then, a perturbation corresponding to the difference between the above two solutions cannot decay in the original open flow model. This contradicts with the assumption of stability [14].

Even though convective instability eventually seems to lead to temporal bifurcations, these may occur only very far downflow. For large $\epsilon$, it was, e.g., verified that the temporal periodicity never becomes very high in a lattice of $10^4$ sites. In a practical sense, therefore, spatial chaos with temporal periodicity may very well occur in a limited region of the lattice. The main difference with the convectively stable case lies in the stability against noise.

In conclusion, we found spatial chaos with temporal periodicity to exist in the one-way coupled logistic lattice. With the help of a newly introduced spatial map which can exactly reproduce the spatial patterns of the spatiotemporal coupled map lattice, it was not only shown that spatial chaos is convectively unstable, but also that stable patterns can exist arbitrarily close to chaos. By employing the concept of comoving Lyapunov exponents, we furthermore found that an important aspect of the bifurcation behavior of the OCLL can be predicted on the basis of the corresponding spatial map.

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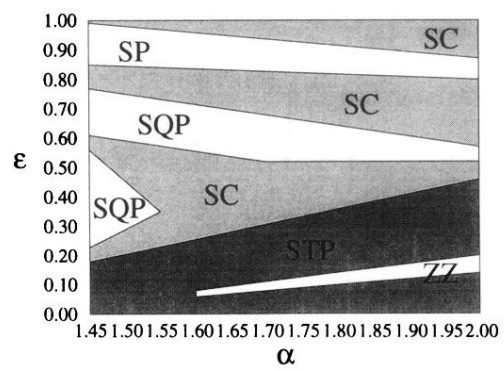
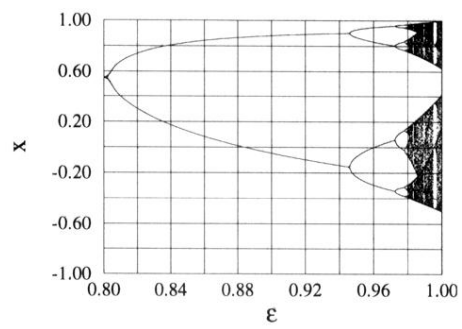FIG. 1. Phase diagram for the open flow model. The system size is $N = 384$.

FIG. 4. Bifurcation cascade of the $k = 1$ spatial map (6). The nonlinearity is $\alpha = 1.5$.