Experimental and Numerical Evidence for Riddled Basins in Coupled Chaotic Systems

James F. Heagy, Thomas L. Carroll, and Louis M. Pecora
Code 6341, Naval Research Laboratory, Washington, DC 20375
(Received 15 August 1994)

We present direct experimental and numerical evidence for riddled basins of attraction for the synchronous chaotic state in a system of coupled chaotic oscillator circuits. Both experiment and computation show scaling typical of basin riddling.

PACS numbers: 05.45.+b, 84.30.Ng

Recent work by Alexander *et al.* [1] has shown that theoretically there can exist chaotic dynamical systems having chaotic attractors that have their basins of attraction literally *riddled* with points from another attractor's basin. Basin riddling frequently occurs when the chaotic attractor exists in an invariant subspace (e.g., a symmetry plane) and the second attractor is not contained within this subspace. The riddling is extreme in that the points of the other basin are *dense* in the chaotic attractor's basin (existing *arbitrarily* close to the chaotic attractor) and have a nonzero measure. These properties imply that any initial condition in the neighborhood of a point going to the chaotic attractor has a nonzero probability of going to the other attractor. Consequently, predictability of the final state is much more difficult than in systems with "ordinary" fractal basin boundaries. Several theoretical and numerical works have appeared that support these findings [2–8]. Lai and Winslow [9,10] have also found that some systems can exhibit riddling in parameter space.

An important issue is whether these effects can be seen in real physical systems. The purpose of this Letter is to address this issue. Ott and Sommerer [4] have conjectured that one place to look for riddled basins is in reaction-diffusion systems of the type $\dot{u} = \nabla^2 u + f(u)$ or their discrete counterparts, diffusively coupled oscillator systems. The latter systems can exhibit synchronous chaotic behavior, and are the systems we consider here.

In the framework of diffusively coupled oscillator systems, the *synchronization manifold* [11], defined simply as the space where all oscillators execute identical dynamics, plays the role of the invariant subspace where the chaotic attractor exists. Motion on it is governed by $\dot{u} = f(u)$ (since $\nabla^2 u = 0$ in the synchronous case). Stability of the synchronous chaotic attractor is determined by the *transverse Lyapunov exponents* [1–7,11]. When the largest of these exponents approaches zero from below there will be points in the synchronization manifold that are dense in the attractor (but of measure zero) and that are repelling from the attractor in directions transverse to the manifold. If there is another (nonsynchronous) attractor in the system, points in the neighborhood of the synchronization manifold can be repelled onto the other attractor. Hence, the second attractor has basin points arbitrarily close to the chaotic attractor—the basin of the synchronous attractor is then said to be *riddled* [12].

Basin riddling is of possible importance in systems where synchronous chaotic behavior has been seen or is expected. There are numerous examples, making a complete list impossible. Representative works are synchronous chaotic lasers [13,14], synchronized chaotic circuits or communications systems [15–17], synchronized, but chaotic biological systems [18,19], and studies using standard chaotic systems coupled in some fashion [20,21].

Recently, Ashwin *et al.* [7] have begun experiments with two coupled chaotic circuits that have shown that near the synchronization threshold there is a "bubbling" behavior of the circuit's motion, suggesting repelling points on the attractor in directions away from the synchronous state. The authors make a case for basin riddling in their system through empirically determined normal Lyapunov exponents obtained from a map constructed from the experimental data. It appears, however, that direct experimental observation of riddled basins and the scaling behavior associated with them [3,5] has not been carried out.

We report here on direct observations of riddled basins in experiments and associated numerical work with a system of synchronized chaotic circuits [22]. The structure of the numerical riddling matches well with the experimental basin picture. Furthermore, both the experimental and numerical results display a scaling of the measure of the basin riddling in accord with existing theory [3,5].

In previous work [11] we showed that diffusively coupled systems with periodic boundary conditions are members of a class of coupled systems with so-called shift symmetry. For N oscillators this symmetry allows one to decompose the behavior near the synchronous state into N spatial Fourier modes from 0 to $N/2$ with 1 through $N/2-1$ being twice degenerate for N even. The 0th mode (the average) is the motion *on* the synchronization manifold with all other modes *transverse* to that manifold. The synchronous state becomes unstable when modes other than the 0th one do not damp out, as determined by the transverse Lyapunov exponents.

Our particular circuit system is a set of four Rössler-like circuits coupled diffusively through their x coordinates

[11]:

$$\dot{x}_i = -fy_i - z_i - rx_i + k(x_{i+1} + x_{i-1} - 2x_i),$$
$$\dot{y}_i = x_i + ay_i, \quad (1)$$
$$\dot{z}_i = -bz_i + cg(x_i),$$

where the ramp function $g(x) = x - d$ if $x > d$ and 0 otherwise, and the indices denoting the particular oscillator node are taken mod4. Typical parameters are $a = 0.13$, $b = 1.0$, $c = 15.0$, $d = 3.0$, $f = 0.5$, and $r = 0.05$. The coupling k varies from 0.0 to 1.0 in our studies.

This system has stable, synchronous chaos for coupling values $0.031 < k < 0.945$ in the model and $0.08 < k < 0.82$ in the circuit. There are four spatial modes for the system: mode 0 (the synchronous state), two mode 1 states ("long wavelength"), and one mode 2 state ("short wavelength"). At $k = 0.82$ in the circuit the synchronous state loses stability with the shortest wavelength mode going unstable first. The same behavior occurs at $k = 0.945$ in the model. We have reported this unusual instability elsewhere [23]. Near the bifurcation threshold (just above and below that value) there are two other attractors off the synchronization manifold which are periodic (period 1) and which have mode 2 spatial components. These nonsynchronous attractors exist in pairs because the oscillators are identical; the attractors are related by a simple shift of indices $i \rightarrow i + 1 \bmod 4$. This is just the situation where we might find riddling and where we do so in this experiment. Figure 1 shows the x-y projection of the three coexisting attractors for the model system at $k = 0.935$.

We can detect (experimentally and numerically) the period-1, nonsynchronous attractors by a simple Fourier analysis of one of the components of the four oscillators [e.g., y_i in Eq. (1)]. The presence of a nonzero mode 2 component allows us to recognize the period-1 attractor(s). A zero mode 2 component implies the system is in the synchronous chaotic state.

We built circuits based on Eq. (1). The circuits are explained in more detail elsewhere [11]. Each circuit was built as an analog computer, and numerical simulations of Eq. (1) confirm that the circuit was well described by these equations. The circuits were coupled in a diffusive fashion, as in the numerical model. The sum $x_{i+1} + x_{i-1} - 2x_i$ was performed by an analog adding circuit. The sum was then fed into an analog multiplier chip, where it was multiplied by a constant voltage which had the same value as the coupling constant k. The coupling constant was fixed just below the short wavelength bifurcation point; in the experiment k was set to 0.805, while in the model k was set to 0.935.

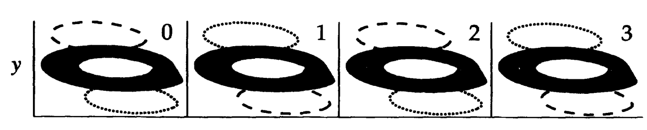

FIG. 1. Projection of three coexisting attractors in the x-y plane for a numerical model of coupled Rössler-like system at $k = 0.935$. The synchronous chaotic attractor is in solid, while the two period-1 attractors are indicated as dashed and dotted lines.

It is convenient in experimental and numerical investigations of riddled basins to construct a two-dimensional grid of initial conditions, with one coordinate representing a direction along the invariant subspace (synchronization manifold) and the other representing a direction transverse to the invariant subspace. At the suggestion of Ott [24], we constructed such a grid for our coupled oscillator system as follows. First, a point (x_S, y_S, z_S) on the synchronous chaotic attractor was selected. The in-sync direction was chosen to be the deviation Δx of the initial x value from x_S for all oscillators. The transverse direction was chosen to have a pure mode 2 Fourier component by setting $y_i = y_s + (-1)^i \Delta y$, $i = 0,1,2,3$. The z coordinate of each oscillator was set to z_S initially. The deviations Δx and Δy were then varied systematically to make up the two-dimensional initial condition grid.

The experiment was computer controlled and automated. Starting from each grid point the circuit was allowed to evolve for 1 s (approximately 1000 times around the attractor) to eliminate transient behavior, at which point the mode 2 Fourier coefficient $S_2 = y_0 - y_1 + y_2 - y_3$ was computed and averaged over the next 100 cycles. If the magnitude of $\langle S_2 \rangle$ fell below a certain empirically determined threshold, the final attractor was labeled as synchronous and a black dot was plotted at the grid point $(\Delta x, \Delta y)$. Otherwise, depending on the sign of $\langle S_2 \rangle$, the attractor was labeled as one of the period-1 attractors and no dot was plotted.

Figures 2(a) and 2(b) show regions of the experimental basins at two magnifications. Figures 2(c) and 2(d) show corresponding basins determined from the numerical model. Each plot shows a 200×200 grid of initial conditions. Only the positive Δy portion is shown in the plots, so that only one period-1 attractor comes into play. The negative Δy portion has the same structure in the numerical model, although because of unavoidable parameter variations in the circuits, the positive and negative portions in the experiment were not mirror images. The basin of the synchronous chaotic attractor (black) shows clear signs of riddling both experimentally and numerically, as evidenced by the number of white dots interspersed throughout the basin. Note that both the experiment and the numerical work show similar structures in the riddling and in the "peaks" present in the synchronous attractor's basin. There is an overall scale difference between the experiment and the numerical simulation in the transverse coordinate Δy. This is most likely due to imprecise parameter matching between the model and the circuit; we have observed the basin plots to be especially sensitive to the tuning of the coupling con-

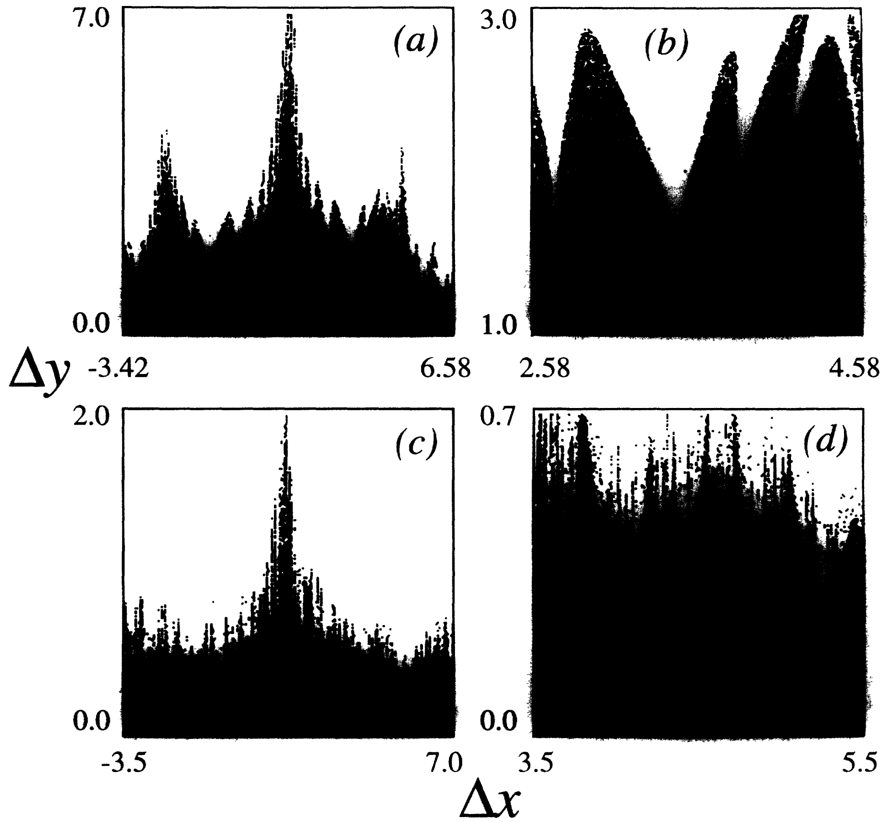

FIG. 2. Basins of attraction for the coupled Rössler-like system in the Δx-Δy plane. Basin of the synchronous chaotic attractor is in black, while basin of period-1 attractor is in white: (a) experimental results, (b) blowup of (a), (c) numerical results, and (d) blowup of (c).

stant k near the bifurcation point. This scale difference is inconsequential as far as the presence of riddling goes.

An important statistic in basin riddling is the measure of the basin riddling as one moves away from the invariant manifold. The measure is taken to be the probability P of going to the period-1 attractor. This probability is predicted to scale as a power law with the transverse coordinate [3,5]; in our case we expect $P \sim \Delta y^{\eta}$. Figure 3(a) shows a log-log plot of P versus Δy for the experimental system. For each line $\Delta y =$ const, the probability P was estimated to be the fraction of points going to the period-1 attractor from a uniformly spaced sample of 1000 points. Figure 3(a) is based on the region shown in Fig. 2(a). Error bars represent 95% confidence intervals in the probabilities P [25]. Also shown in the plot is the best fit power law to the scaling region of the data. Each point in the fit is weighted by the inverse of its accompanying error bar. Figure 3(b) shows a similar plot for the numerical model based on the region shown in Fig. 2(c). Power law scaling with a critical exponent near $\eta = 2$ is seen in the experiment and the numerical model.

We consider Figs. 2 and 3 to provide compelling evidence for the existence of basin riddling in this system. These results appear not to be specific to our system, in the sense that they require only a synchronous chaotic state and attractors off the synchronization manifold near the synchronization threshold. Many coupled, chaotic systems should fulfill these requirements. This corroborates the conjecture by Ott and Sommerer [4] that basin riddling should be common in such systems.

Recent work [5] has shown that the effects of noise can be incorporated within the basin riddling theory. Although we have not investigated these effects, we can conclude that a small amount of noise (invariably present in any electronic circuit system) does not appear to eliminate the basin riddling phenomenon. For small transverse deviations noise effects do appear in the experimental probability plot [Fig. 3(a)]. As $\Delta y \rightarrow 0$ the probability of going to the nonsynchronous attractor appears to level off instead of displaying power law scaling behavior. This suggests that noise effects become dominant near the synchronization manifold.

There are some suggestions that coupled systems operating in a synchronous chaotic fashion might be of use technologically. These uses include lasers [13,14] and circuits [26–29]. The significant effect of riddled basins in coupled systems strongly suggests that such applications

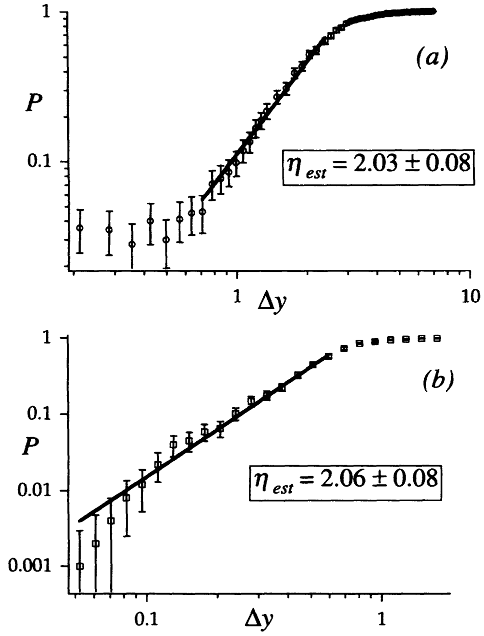

FIG. 3. Log-log plot of probability P of going to the period-1 attractor as a function of the mode 2 Fourier amplitude Δy: (a) experimental results; (b) numerical results. Power law fits to the scaling regions in each plot are carried out by weighting each point according to the inverse of its error bar.

should have operating points away from synchronization thresholds. In the few systems we have tested so far, we have found that riddled basins are avoided when this is done.

JFH gratefully acknowledges support from the National Research Council and ONR.

[1] J.C. Alexander, J.A. Yorke, Z. You, and I. Kan, Int. J. Bifurcations and Chaos **2**, 795 (1992).
[2] J.C. Sommerer and E. Ott, Nature (London) **365**, 138 (1993).
[3] E. Ott, J.C. Sommerer, J.C. Alexander, I. Kan, and J.A. Yorke, Phys. Rev. Lett. **71**, 41 (1993).
[4] E. Ott and J.C. Sommerer, Phys. Lett. **A 188**, 39 (1994).
[5] E. Ott, J.C. Alexander, I. Kan, J.C. Sommerer, and J.A. Yorke, Physica (Amsterdam) **76D**, 384 (1994).
[6] I. Kan, Bull. Am. Math. Soc. **31**, 68 (1994).
[7] P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. **A 193**, 126 (1994).
[8] R.H. Parmenter and L.Y. Yu, Phys. Lett. **A 189**, 181 (1994).
[9] Y.-C. Lai and R.L. Winslow, Phys. Rev. Lett. **72**, 1640 (1994).
[10] Y.-C. Lai and R.L. Winslow, Physica (Amsterdam) **74D**, 353 (1994).
[11] J.F. Heagy, T.L. Carroll, and L.M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[12] These properties call into question the very definition of an attractor. All points on an attractor are often assumed to be contained in some small enough open set consisting only of points from that attractor's basin. This is no longer the case when the basin is riddled. Several researchers have therefore suggested adopting the definition of attractor proposed by J. Milnor [Commun. Math. Phys. **99**, 177 (1985)], which, roughly speaking, is a set which has a dense orbit and which attracts a nonzero phase space volume of points.
[13] H.G. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990).
[14] R. Roy and K.S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994).
[15] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[16] K. Cuomo and A.V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993).
[17] N.F. Rul'kov, A.R. Volkovskii, A. Rodriguez-Lozano, E.d. Rio, and M.G. Velarde, Int. J. Bifurcations and Chaos **2**, 669 (1992).
[18] E.E. Shnöl, Prikl. Matem. Mekhan. U.S.S.R. **51**, 9 (1987).
[19] S. Bressler, R. Coppola, and R. Nakamura, in Proceedings of the 2nd Experimental Chaos Conference, Arlington, 1993 (World Scientific, Singapore, to be published).
[20] J.M. Kowalski, G.L. Albert, and G.W. Gross, Phys. Rev. A **42**, 6260 (1990).
[21] V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **29**, 795 (1986) [Radiophys. Quantum Electron. **29**, 795 (1986)].
[22] The use of analog circuits in experiments is widespread. In addition to Ref. [7], we refer the reader to *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P.V.E. McClintock (Cambridge University Press, Cambridge, 1989), Vols. 1–3.
[23] J.F. Heagy, L.M. Pecora, and T.L. Carroll (to be published).
[24] E. Ott (private communication).
[25] R.E. Wapole and R.H. Myers, *Probability and Statistics for Engineers and Scientists* (Macmillan, New York, 1972).
[26] K.M. Cuomo, Int. J. Bifurcations and Chaos **3**, 1327 (1993).
[27] J. Gullicksen *et al.,* in *Proceedings of the First Experimental Chaos Conference, Arlington, 1991* (World Scientific, Singapore, 1992).
[28] L.M. Pecora, in *Chaos in Communications,* SPIE Proceedings (SPIE–The International Society for Optical Engineering, Bellingham, WA, 1993), pp. 2–25.
[29] N.F. Rul'kov and A.R. Volkovskii, in *Chaos in Communications,* SPIE Proceedings (SPIE–The International Society for Optical Engineering, Bellingham, WA, 1993), pp. 132–140.

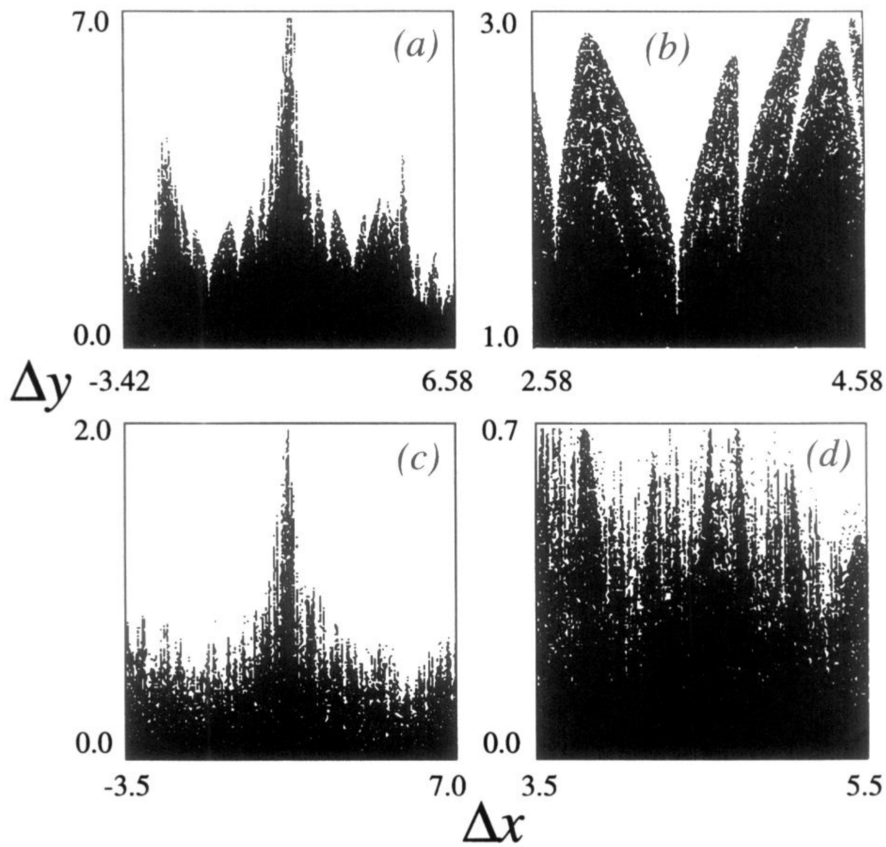

FIG. 2. Basins of attraction for the coupled Rössler-like system in the Δx-Δy plane. Basin of the synchronous chaotic attractor is in black, while basin of period-1 attractor is in white: (a) experimental results, (b) blowup of (a), (c) numerical results, and (d) blowup of (c).