Experimental Observation of Berry's Phase of the Lorentz Group

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Berry's phase of the Lorentz group has been measured in a system of connected nonlinear electronic oscillators. Each oscillator was driven so that it was in the vicinity of a period-doubling bifurcation where the response to a small perturbation transforms similarly to a Lorentz boost of a spinor. By connecting the output of one oscillator to the input of the next, it was possible to make a closed loop on a hyperboloid (the invariant surface of the Lorentz group), the enclosed area of which is Berry's phase. Good agreement between theory and experiment is found.

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In recent years there has been considerable interest [1–4] in the so-called geometric phases, in both classical and quantum mechanics. A geometric phase is time independent but path dependent; i.e., it is a consequence of the geometry of the trajectory, but not a function of the dynamics of the motion. These phases approach a finite nonzero limit as the system is taken infinitely slowly around a closed path. The concept has had wide-ranging implications. It has been applied in fields as diverse as hydrodynamics [5], optics [6], nuclear magnetic resonance [7–9], the quantized Hall effect [10], and quantum field theory [11]. It is a simple, subtle, and mathematically beautiful concept.

In this work, we have measured the geometric phase in a system of connected dissipative driven nonlinear electronic oscillators in the vicinity of a period-doubling bifurcation. Surprisingly, it turns out that under such conditions a small signal is transformed as a Lorentz transformation of a spinor, except for some factors [12]. By connecting the output of one oscillator to the input of the next, it is possible to make what corresponds to a closed loop on a hyperboloid (the invariant surface of the Lorentz group), and thereby measure Berry's phase of the Lorentz group (LG). The theory was originally formulated at the transition between a dissipative system and a Hamiltonian system [12,13]. However, such a limit is difficult to obtain in an experimental situation, and it has therefore been necessary to include the effects of dissipation. Unexpectedly, such a correction does not destroy the above transformation properties, but merely gives some additional amplitude dependent phase factors that can be removed by a renormalization. Usually when speaking of the LG one thinks of the special theory of relativity, but the LG is a mathematical group and one should not be surprised to find it in other contexts. Chiao and Jordan [14,15] first suggested an experiment to measure Berry's phase of the LG using squeezed light. The connection between squeezed light, the aforementioned oscillators, and the LG lies in the transformation properties of such systems in the vicinity of an instability [12]. In the present study, the dissipative nature of the system is of the utmost importance and is responsible for the stability of the transformation.

A dissipative driven nonlinear oscillator can be described by a second-order differential equation of the form

$$\ddot{q} + \alpha\dot{q} + V'(q) = A_D\cos(2\omega_R t + \phi) + A_S\cos(\omega_R t + \theta), \qquad (1)$$

where q is some generalized coordinate and α is the damping parameter. $V(q)$ is a nonlinear potential with $V''(0) = \omega_0^2$. This means that the system has a small amplitude resonant frequency given by $\omega_R^2 = \omega_0^2 - \alpha^2/2$. The important nonlinear part of the potential is given by the third-order derivative $V'''(0) = \gamma$. [Higher-order terms are unimportant since the bifurcation point and therefore the unperturbed ($A_S = 0$) limit cycle $q_0(t)$ tend to zero as $\alpha \to 0$, which is the case studied here. See Ref. [13] for details.] The first term on the right-hand side of Eq. (1) is the drive with amplitude A_D, frequency $2\omega_R$, and phase ϕ. The second term is a small perturbation with amplitude A_S, frequency ω_R, and phase θ. It is the system's response to this perturbation which will be studied. The basic assumption is that as the amplitude A_D of the drive is increased, the system will eventually undergo a period-doubling bifurcation at a critical amplitude A_C. Since it is not possible to do the experiment in the scaling limit $\alpha \to 0$, the theory [12] has been extended to include the lowest-order correction term in a perturbation expansion in the damping parameter α.

Linearizing Eq. (1) around the limit cycle $q_0(t)$, we find the solution

$$\xi(t) = \frac{A_S}{\alpha\omega_R}[\sin(\omega_R t + \theta) + u\sin(\omega_R t + 2\phi - \theta) + \alpha(1 - u^2)/4\omega_R]/(1 - u^2), \qquad (2)$$

where $\xi(t) = q(t) - q_0(t)$ is the deviation from the limit cycle $q_0(t) = (-A_D/3\omega_R^2)\cos(2\omega_R t + \phi)$, and $u = A_D/A_C$ where $A_C = 6\alpha V''(0)^{3/2}/V'''(0)$. Transforming to a rotating frame of reference (rotating with the

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signal frequency ω_R), the response can be written as a linear transformation of a spinor, namely,

$$\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = a(u)\Lambda(u,\phi)\boldsymbol{R}(\delta)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad (3)$$

where $a(u) = a_0/(1-u^2)^{1/2}$ with $a_0 = A_S/\alpha\omega_R$,

$$\Lambda(u,\phi) = \frac{-\sigma_0 - u\sigma_x\cos(\phi+\delta) - u\sigma_z\sin(\phi+\delta)}{(1-u^2)^{1/2}}, \quad (4)$$

where $\delta = \tan^{-1}[Q/(1-u^2)]$ and $Q = 4\omega_R/\alpha$, and the pure rotations are $\boldsymbol{R}(\delta) = \sigma_0\cos\delta + i\sigma_y\sin\delta$ where $(\sigma_0, \sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices given by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

That this is indeed similar to a Lorentz transformation of a spinor (except for some factors) can be seen with the parametrization

$$\cosh(\Phi/2) = 1/\sqrt{1-u^2}, \quad \sinh(\Phi/2) = u/\sqrt{1-u^2}, \quad (5)$$

so that $\Lambda(u,\phi)$ becomes

$$\Lambda(\boldsymbol{\Phi}) = \exp\left(-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}}{2}\right), \quad (6)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\boldsymbol{\Phi} = \Phi(\cos(\phi+\delta), 0, \sin(\phi+\delta))$, and Φ is obtained from Eq. (5). Thus, Eq. (6) is similar to a Lorentz transformation of a spinor where the direction of the boost is determined by the phase of the driving field and the dissipation in the system [16].

The aim of this work is to measure Berry's phase of the LG. In order to do this a number of successive transformations have to be combined in such a way that they make a closed loop on a hyperboloid (the invariant surface of the LG). For example, four successive transformations of the form Eq. (4) can be combined to yield [ignoring for the moment the other factors in Eq. (3)]

$$\boldsymbol{L}(u,\phi+\delta) = \Lambda(u,\pi/4)\Lambda(u,\phi+\delta) \times\Lambda(u,2\pi-\phi-\delta)\Lambda(u,-\pi/4). \quad (7)$$

Notice that the normalized amplitude u is the same for all four "boosts" as is the Q factor. This ensures that δ is the same for all the boost directions and can therefore be ignored. The above transformation makes a closed loop on the hyperboloid (see Fig. 1) if the following constraint is fulfilled:

$$g(\phi) = 1 - \sqrt{2}\cos\phi + u^2[\sqrt{2}\cos\phi + \cos 2\phi + \sin 2\phi] = 0. \quad (8)$$

In the following, we will use the branch of the solution which runs continuously from $\phi = 3\pi/4$ for $u \approx 0$ to $\phi = \pi/2$ for $u = 1$. Notice for $u \approx 0$ the sum of the angles is equal to 2π, whereas for $u \approx 1$ the sum is close to $5\pi/2$. The latter is a consequence of the non-Euclidean geometry.

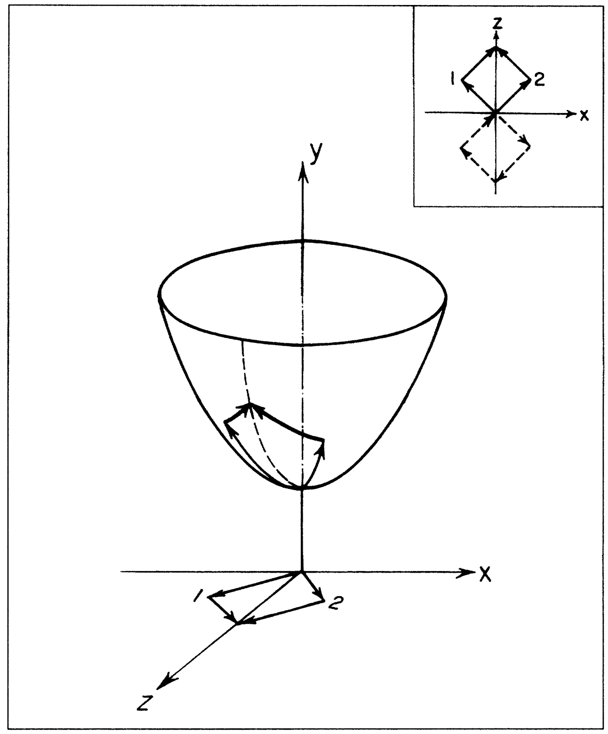

FIG. 1. View of the closed path on the hyperboloid with the projection into the parameter space. The coordinate system corresponds to Eq. (4). Each vector corresponds to a Lorentz boost, with a total of four to complete the closed loop. The inset shows the difference between using four successive boosts (dashed vectors) as in Eq. (7), and a pair of two successive boosts (solid vectors) in opposite directions as in Eqs. (12) to close a loop. The enclosed area is Berry's phase θ_B.

The initial state is given by $\boldsymbol{\xi}_i = \boldsymbol{R}(\theta)(1,0)$ where $\boldsymbol{R}(\theta)$ rotates a spinor through an angle θ. As a result of the transformation Eq. (7) and the constraint Eq. (8), the final state becomes

$$\boldsymbol{\xi}_f = \boldsymbol{L}(u,\phi)\boldsymbol{\xi}_i = \boldsymbol{R}(\theta_B)\boldsymbol{R}(\theta)(1,0) = \boldsymbol{R}(\theta_B)\boldsymbol{\xi}_i, \quad (9)$$

where the rotation $\boldsymbol{R}(\theta_B)$ is a result of the closed loop, and $\theta_B(u)$ is Berry's phase with the geometric origin given by

$$\theta_B(u) = \int_{\Omega(u)} d\Omega, \quad (10)$$

where the integral is over the area (Lorentz metric) on the hyperboloid enclosed by the loop. The angle $\theta_B(u)$ is more easily calculated using the Pauli spin matrix representation of the LG.

Now to the particular form of the transformation used in the experiment. The transformation $\boldsymbol{L}(u,\phi)$ involves four operations and therefore four systems. It is desirable to use only two systems at one time. From Eq. (9), we have for a closed path that $L(u,\phi) = R(\theta_B)$, which, using Eq. (7), can be rearranged into

$$\Lambda(u,\phi)\Lambda(u,\pi/4) = \Lambda(u,\pi-\phi)\Lambda(u,3\pi/4)R(\theta_B). \tag{11}$$

Thus, we can consider the closed loop formed by the two transformations,

$$\Lambda(u,\phi)\Lambda(u,\pi/4) = \Lambda(u^*,\phi^*)R(-\theta_B/2), \tag{12a}$$

$$\Lambda(u,\pi-\phi)\Lambda(u,3\pi/4) = \Lambda(u^*,\phi^*)R(\theta_B/2). \tag{12b}$$

The parameters u^* and ϕ^* are defined through the above relations. The idea is to make two measurements with two systems in what corresponds to an interference experiment, along the lines suggested by Chiao and Jordan [14]. Thus, instead of having four individual paths traversing a closed loop in one direction (the dashed vectors in the inset to Fig. 1), the same loop is traversed in two directions (the solid vectors). If now the full form of the transformation used in the experiment is considered, the additional factors from Eq. (3) have to be included, yielding

$$L_1(u,\phi) = a^2(u)\Lambda(u,\phi)R(\delta)R(\delta_a)\Lambda(u,\pi/4)R(\delta), \tag{13a}$$

$$L_2(u,\phi) = a^2(u)\Lambda(u,\pi-\phi)R(\delta)R(\delta_a)\Lambda(u,3\pi/4)R(\delta), \tag{13b}$$

for paths 1 and 2, respectively (see Fig. 1). The rotation $R(\delta)$ does not commute with the Lorentz boosts, and has to be removed. This is done experimentally by correcting with a phase shift δ_a such that $R(\delta)R(\delta_a) = \sigma_0$. Assuming this constraint and using Eqs. (12), the transformations Eqs. (13) can be written as

$$L_1(u,\phi) = a^2(u)\Lambda(u^*,\phi^*)R(\delta-\theta_B/2), \tag{14a}$$

$$L_2(u,\phi) = a^2(u)\Lambda(u^*,\phi^*)R(\delta+\theta_B/2). \tag{14b}$$

These transformations involve a stretching, a contraction, and a rotation (the points on a circle are rotated and transformed into an ellipse). Therefore, by varying the phase θ of the perturbing signal in Eq. (1), the component of the response $\xi(t)$ at the frequency ω_R goes through a minimum and maximum corresponding to the minor and major axes of the ellipse. Berry's phase will then be the difference between the phases $\theta_{\min}$ corresponding to the minimum for each of the two paths in Eqs. (14), i.e., $\theta_B = \theta^1_{\min} - \theta^2_{\min}$.

A schematic of the experiment is shown in Fig. 2. At the heart of it are two nominally identical nonlinear oscillators shown in the inset. The nonlinearity is obtained by using a diode (BB212) in the oscillator section whose capacitance $C(V)$ is a function of voltage. There are isolating amplifiers both before and after the oscillator section. The inductor L_0 and resistor R_0 were adjusted so that both oscillators had the same resonant frequency ($f_R = \omega_R/2\pi =$ 50 kHz) and damping ($Q = 14.7 \pm 0.4$). In principle we would like as large a Q as possible, but it was found in practice that the measurement became too sensitive for larger values. However, the Q was large enough so that it was possible to use the theory with the damping corrections discussed at the beginning of the paper.

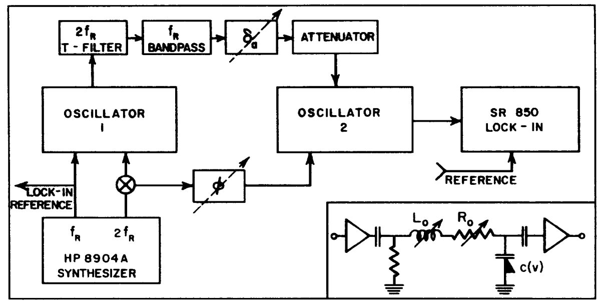

FIG. 2. A schematic of the experiment (see text). Inset: a schematic of one of the nonlinear oscillators.

The oscillators were driven by a Hewlett-Packard HP8904A frequency synthesizer which provided both f_R and $2f_R$ and control over the phase θ. The perturbing amplitude A_S was set at 1 mV which was small enough so the theory would be valid. The drive amplitude A_D was typically 200 mV. It was adjusted so that it was normalized relative to the (identically adjusted) bifurcation points of the oscillators. The drive frequency was split and connected directly to oscillator 1 and phase shifted by ϕ before entering oscillator 2 [i.e., $\phi = 0$ for oscillator 1; see Eq. (1)]. Therefore, the phase ϕ, determined by the constraint Eq. (8) and Eqs. (12), selected either path 1 or path 2 (see Fig. 1).

The perturbing signal was fed into oscillator 1. Since we are interested only in the signal at f_R, the output from oscillator 1 was first notch filtered at $2f_R$ to strongly suppress that component and then bandpass filtered at f_R to remove the higher harmonics produced by the nonlinearity of the oscillator. As discussed earlier, an additional rotation $R(\delta_a)$ was included since there must be no phase rotation from the output of oscillator 1 to the input of oscillator 2. Then, since the signal at f_R is amplified by oscillator 1, it must be attenuated back to its original small amplitude before entering oscillator 2. Phase shifts were measured with a Stanford Research SR850 lock-in amplifier with a 10 MΩ probe. (The insertion of the probe affected the impedance of the circuit under measurement, so there was always an uncertainty of $\lesssim 1\%$ in any phase measurement.) Finally, the output from oscillator 2 was measured with the lock-in amplifier and the amplitude of the signal at f_R recorded as a function of the phase shift θ. The value $\theta_{\min}(u)$ which corresponded to a minimum in the amplitude was recorded together with the value of the normalized amplitude u. This procedure was repeated for both paths 1 and 2. The difference of these phases yields the Berry's phase, i.e., $\theta_B(u) = \theta^1_{\min}(u) - \theta^2_{\min}(u)$ which is shown in Fig. 3. It is seen

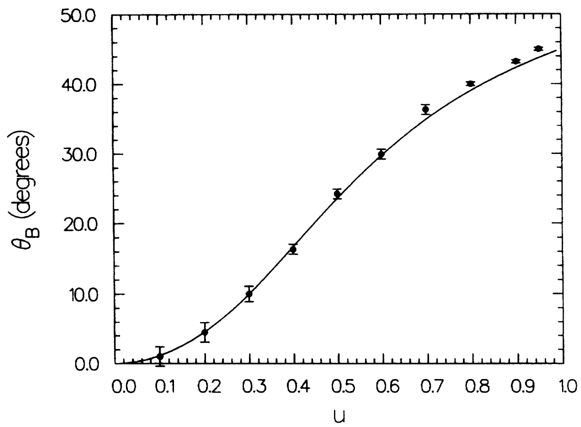

FIG. 3. Berry's phase θ_B as a function of the normalized driving amplitude $u = A_D/A_C$. The solid line is the theoretical curve computed from Eq. (10).

that the agreement between the measured and theoretical values is very good.

One should note that the experiment was not performed merely to simulate the Lorentz group. Rather, it is an example of a large class of nonlinear systems which, when tuned in the vicinity of an instability, show, surprisingly, a response similar to a Lorentz boost. It is interesting to note the similarity between the above nonlinear theory in which dissipation is an important part and optical phase squeezed states [17].

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