Superluminality, Parelectricity, and Earnshaw's Theorem in Media with Inverted Populations

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(Received 25 August 1994)

Population inversion leads to the phenomena of *superluminality,* in which wave packets can propagate dispersionlessly near dc faster than c, and *parelectricity,* in which the dc susceptibility is negative, implying the levitation of a charge above a parelectric medium. Also, charges placed in a cavity surrounded by a parelectric medium form stable configurations, in seeming violation of Earnshaw's theorem. Einstein causality is not violated.

PACS numbers: 41.20.Cv, 41.20.Jb, 42.25.Bs, 42.52.+x

Since the pioneering work of Gordon, Zeiger, and Townes on the ammonia maser forty years ago [1], it has been well known that the inversion of populations of atomic and molecular energy levels leads to stimulated emission, and hence amplification, of electromagnetic radiation. However, it is not well known that the same inversion of populations also leads in principle to superluminal propagation of wave packets, whose group velocity near dc can exceed c without violating Einstein causality [2]. It is also not well known that population inversion, which implies the existence of parelectric media with negative dc electric susceptibilities, leads to surprising consequences in electrostatics. (*Parelectric* media should not be confused with *paraelectric* media, which are ferroelectrics just above their Curie points.) We show here that the existence of a parelectric medium implies the possibility of levitation of a test charge in the vacuum above this medium, as well as stable electrostatic configurations of charges placed inside an evacuated cavity surrounded by this medium. The resulting apparent violation of Earnshaw's theorem will be discussed.

We shall begin by considering a concrete example of a medium with inverted populations. Because of tunneling, the nitrogen atom in the ammonia molecule, NH_3, can be either above the plane of the hydrogens, or below it. This leads to a splitting (approximately 24 GHz) of the $K \neq 0$ rotational levels of the molecule, in which the lower energy state $|+\rangle$ is the symmetric linear combination of the two possible configurations of the nitrogen atom, and the upper state $|-\rangle$ is the antisymmetric linear combination of these two possibilities. Upon the application of a dc electric field, the two energy eigenvalues associated with this doublet repel away from each other. The energy level shifts can be readily calculated by means of second order perturbation theory, yielding for the upper and lower states, respectively,

$$\Delta W_- = +\tfrac{1}{2}|\alpha|\mathcal{E}^2 > 0 \quad \text{and} \quad \Delta W_+ = -\tfrac{1}{2}|\alpha|\mathcal{E}^2 < 0\,, \tag{1}$$

where $\mathcal{E}$ is the dc electric field strength, and $|\alpha|$ is a positive constant. Now the change in energy of a small sphere of polarizability α placed in an applied dc electric field $\mathcal{E}$ is given by

$$\Delta W = -\int_0^{\mathcal{E}} \mathbf{p}\cdot d\mathbf{E} = -\frac{1}{2}\alpha\mathcal{E}^2\,, \tag{2}$$

where the induced dipole moment $\mathbf{p}$ is related to the inducing electric field $\mathbf{E}$ by $\mathbf{p} = \alpha\mathbf{E}$. Hence the polarizability of molecules in the lower state, $\alpha_+ = +|\alpha| > 0$, is *positive,* whereas the polarizability of molecules in the upper state $\alpha_- = -|\alpha| < 0$ is *negative.* One immediate consequence of this is that the dc susceptibility $\chi(0) = -N|\alpha|$ of ammonia gas with all its molecules in the upper state is *negative;* i.e., this medium is *parelectric.* (We neglect local field corrections, although these become important in parelectric condensed matter.) Such a medium is in a metastable, but not thermodynamic, equilibrium. The *imaginary* part of the susceptibility of this inverted medium undergoes a sign reversal with respect to that for an uninverted medium, thus leading to microwave amplification by stimulated emission of radiation. The *real* part of the susceptibility of this inverted medium, which also undergoes a sign reversal with respect to that for an uninverted medium, can lead to striking new phenomena, such as superluminality and parelectricity.

We shall first discuss the phenomenon of superluminality. The refractive index of a completely inverted two-level medium is given by

$$n(\omega) = \left(1 - \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)^{1/2}\,, \tag{3}$$

where γ is a (small) phenomenological linewidth, ω_0 is the resonance frequency of the medium, and $\omega_p = (4\pi Ne^2|f|/m)^{1/2}$ is the effective plasma frequency, where N is the number density of atoms or molecules in the upper state, e is the electron charge, $|f| = 2m\omega_0|\langle -|x|+\rangle|^2/\hbar$ is the absolute value of the oscillator strength of the transition, and m is the electron mass. (Typically, $\gamma \ll \omega_p \ll \omega_0$.) Note that the minus sign in front of the second term under the square root in Eq. (3) arises from complete population inversion: It differs from its usual positive sign for an uninverted medium. As a result of this sign change, the index of refraction near zero frequency is less than unity,

0031-9007/94/73(25)/3383(4)$06.00

$n(0) = (1 - \omega_p^2/\omega_0^2)^{1/2} < 1$. From Eq. (3) it also follows that the slope $d\{\mathrm{Re}\, n(\omega)\}/d\omega$ approaches zero as $\omega \to 0$. Hence the medium is dispersionless near dc.

Now let a classical, finite-bandwidth wave packet, whose carrier frequency $\omega_0/2\pi$ (say around 1 GHz) and spectrum lie far below the resonance frequency $\omega_0/2\pi = 24$ GHz of the ammonia molecules, be incident upon a medium consisting of a gas of these molecules prepared entirely in the upper state. The amplitude of this wave packet will be chosen sufficiently small so that only the *linear* response of the medium to this weak perturbation need be considered. The fact that $n(0) < 1$ implies that the phase velocity, $v_p(0) = c/n(0) > c$, is greater than the vacuum speed of light c. It is well known that phase velocities can exceed c without any violation of relativity. However, more surprisingly, here near zero frequency the group velocity,

$$\begin{aligned} v_g(0) &= \left(\frac{d\mathrm{Re}\, k(\omega)}{d\omega}\right)^{-1}_{\omega_c \to 0} \\ &= c\left[\mathrm{Re}\, n(\omega) + \omega \frac{d\mathrm{Re}\, n}{d\omega}\right]^{-1}_{\omega_c \to 0} \\ &= \frac{c}{n(0)} > c\,, \end{aligned} \tag{4}$$

is equal to the phase velocity, and therefore also exceeds c. Furthermore, in contrast to a medium in its ground state, since this inverted ammonia medium can temporarily loan part of its stored energy to the forward tail of the wave packet, the energy velocity [3], which is the velocity of energy transport, is also superluminal

$$v_E(0) \equiv \frac{\langle S\rangle}{\langle u\rangle} = \frac{c}{\sqrt{\epsilon(0)}} = \frac{c}{n(0)} > c\,, \tag{5}$$

where $\langle S\rangle$ is the time-averaged Poynting vector, $\langle u\rangle$ is the time-averaged energy density, and $\epsilon(0)$ is the zero-frequency dielectric constant. Energetically, a medium in its ground state requires an initial transfer of energy from the wave to the medium in order to polarize it, thus leading to a *retarded* transmitted wave packet. However, when the medium possesses an inverted population, energy is stored in the medium, thus allowing an initial transfer of energy from the medium to the wave [4]. The transmitted wave packet can now be *advanced* rather than *retarded,* thus leading to the possibility of superluminal propagation.

All of the above wave velocities are equal to each other because this medium with inverted populations is essentially transparent and dispersionless near zero frequency, as it is for any dielectric medium near $\omega = 0$. Thus, dispersionless superluminal propagation occurs in a wide spectral window stretching from dc to the low-frequency side of resonance. By Fourier's theorem, any arbitrary wave packet or wave form should travel through this medium superluminally without appreciable change in amplitude or shape, provided that its total bandwidth is finite and lies far below resonance. This superluminal propagation can be thought of as a causal wave-form-reshaping process in which the weak forward tails of the wave form undergo *virtual* amplification by the inverted medium, followed by *virtual* absorption of the peaks of the incident wave form. This produces an *advanced* wave form (versus a *retarded* wave form produced in an ordinary medium), which faithfully reproduces the entire incident wave form, no matter how complex.

A more general way to demonstrate these results is to start from the Kramers-Kronig relations for the complex electric susceptibility $\chi(\omega)$, from which one can derive the zero-frequency sum rule (Sec. 62 of [5])

$$\chi(0) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\chi(\omega)}{\omega}d\omega\,, \tag{6}$$

where $\chi(0)$ is the (real) zero-frequency susceptibility. In the case of the ammonia gas with inverted populations, $\mathrm{Im}\,\chi(\omega) < 0$ in the range of frequencies near the strong low-frequency resonance at 24 GHz, which gives the dominant contribution to the integral in Eq. (6). From this it follows that the dc electric susceptibility $\chi(0)$ is negative, or equivalently, that the dc dielectric constant $\epsilon(0) = 1 + 4\pi\chi(0) < 1$ is less than unity. Thus, the medium is parelectric. It also follows that the index of refraction near zero frequency,

$$n(0) = \epsilon(0)^{1/2} = [1 + 4\pi\chi(0)]^{1/2} < 1\,, \tag{7}$$

is also less than unity. Since the medium is dispersionless near dc (a fact that also follows from the Kramers-Kronig relations; see Sec. 64 of [5]), the phase, group, and energy velocities all exceed the vacuum speed of light c. Since we have now proved that both parelectricity and superluminality follow from the Kramers-Kronig relations, these results do not depend on the validity of any specific model such as the Lorentz model, or the two-level model. Also, since causality was used to derive these relations, causality cannot be violated by these conclusions. In general, the Kramers-Kronig relations imply that any medium with gain gives rise to superluminal group velocities in a transparent spectral window separate from the region with gain [2,6,7].

Einstein causality is not violated by these conclusions, because there is no information contained in the peaks of the wave packet or wave form which is not already present in its weak forward tails. New information is communicated only when there is an unexpected change, e.g., a discontinuity in the wave form, whose arrival time cannot be inferred from its past behavior. A simple example of such a discontinuity is that of a step-modulated sine wave, which Sommerfeld and Brillouin used as their incident wave form in their study of precursors [3]. Such discontinuous wave forms, in contrast to the smooth, finite-bandwidth wave packets or wave forms which we consider here, contain components at infinite frequency, so that it is the infinite-frequency index of refraction, $n(\infty) = 1$, that determines the propagation speed of the discontinuity. Thus the propagation speed of discontinuous signals is exactly c.

It is helpful at this point to review Sommerfeld and Brillouin's definitions of various wave velocities [3]. For an absorptive, dispersive medium, they found it necessary to define five different kinds of wave velocities: (1) the *phase* velocity, at which the zero crossings of the carrier wave would move; (2) the *group* velocity, at which the peak of a wave packet would move; (3) the *energy* velocity, at which energy would be transported by the wave; (4) the *"signal"* velocity, at which the half-maximum wave amplitude would move; and (5) the *front* velocity, at which the first appearance of the discontinuity would move. All these five velocities differed from each other in the region of anomalous dispersion near an absorption line. The first two kinds of wave velocities, the phase and group velocities, might be superluminal (in fact, they might become infinite, or even negative), but the next two kinds, the energy and signal velocities, were subluminal under all the circumstances they considered. The last kind, the front velocity, was equal to the vacuum speed of light c.

However, their "signal" velocity should not be confused with the information velocity of special relativity, which must be strictly less than or equal to c. We have shown that in the case of a medium with inverted populations, due to its dispersionlessness, all the first four kinds of wave velocities (the phase, group, energy, and signal velocities), when extended to the low-frequency, finite-bandwidth wave packets considered here, are superluminal, and that only the fifth (the front velocity) is not. We have identified the front velocity as the information velocity of special relativity. Thus there is no violation of Einstein causality.

In the above classical analysis, we have ignored issues of signal-to-noise ratios. Quantum noise associated with spontaneous emission, however, should not affect these conclusions [4]. The effect of Johnson noise has not been taken into account here, but we do not expect it to change these results.

We shall now discuss the consequences of parelectricity. In analogy with magnetism, where diamagnetism and paramagnetism correspond to the screening and antiscreening, respectively, of an applied dc magnetic field by media with opposite signs of the magnetic susceptibility, here we can have the phenomena of dielectricity and *parelectricity,* which correspond to the screening and antiscreening of an applied dc electric field, respectively, by media with opposite signs of the electric susceptibility. In the usual textbook discussions of dielectrics, the possibility of parelectric media is either not discussed at all, or dismissed as being unphysical on the basis of equilibrium thermodynamic arguments (Secs. 14 and 61 of [5]), which by their nature do not encompass the possibility of metastable, inverted media.

Parelectric media can lead to striking new phenomena in addition to superluminality. In analogy with a small ferromagnet which can be levitated above a superconducting surface, a small ferroelectric body can, in principle, be levitated above a strongly parelectric surface. The origin of this phenomenon lies in the fact that the image charge $q_{\rm image}$, induced by a test charge $q_{\rm test}$ placed in the vacuum above a planar surface of a dielectric medium, is given by

$$q_{\rm image} = -\frac{\epsilon - 1}{\epsilon + 1} q_{\rm test} , \tag{8}$$

where $\epsilon \equiv \epsilon(0)$ is the dc dielectric constant. In the case of ordinary dielectrics with $\epsilon > 1$, the image charge has the *opposite* sign from the test charge, and this fact leads to mechanical instability: The test charge is unstable due to the increasing mutual attraction toward its image, as it approaches the surface. However, in the case of parelectrics with $\epsilon < 1$, there is a qualitative change in the behavior of the system. Now the image charge has the *same* sign as the test charge. This leads to a mutual *repulsion* between the test charge and its image charge. Levitation of the test charge above a parelectric medium results. The equilibrium height h of the levitated test charge above the surface of the parelectric is given by

$$h = \frac{1}{2} q_{\rm test} \left[\left(\frac{1 - \epsilon}{1 + \epsilon} \right) \frac{1}{mg} \right]^{1/2} , \tag{9}$$

where m is the mass of the test charge, and g is the acceleration due the Earth's gravity. An infinitesimal displacement in the vertical direction dh results in a Hooke's restoring-force law

$$dF = -\frac{4}{q_{\rm test}} (mg)^{3/2} \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{1/2} dh , \tag{10}$$

implying stable equilibrium in the vertical direction for $\epsilon < 1$. Neutral stability, however, holds in the two horizontal directions. Small vertical displacements from equilibrium result in simple harmonic motion at an angular frequency

$$\Omega_{\rm vertical} = 2\, q_{\rm test}^{-1/2}\, g^{3/4}\, m^{1/4} \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{1/4} . \tag{11}$$

Absolute stability for all directions of displacement can be achieved by levitating the test charge above a bowl-shaped parelectric medium.

We have also solved the problem of the mechanical stability of a test charge placed at the center of an evacuated spherical cavity of radius a surrounded by a uniform parelectric medium, in the absence of gravity. The force experienced by the test charge after it is displaced from the center by an infinitesimal displacement $d\mathbf{r}$ is given by

$$d\mathbf{F} = -\frac{1 - \epsilon}{1 + 2\epsilon} \frac{2 q_{\rm test}^2}{a^3} d\mathbf{r} . \tag{12}$$

This again has the form of Hooke's law, provided that $\epsilon < 1$, i.e., that the medium surrounding the spherical cavity is parelectric. (In the case of ordinary dielectrics where $\epsilon > 1$, the equilibrium at the center is unstable.) This solution was obtained both by using the method of separation of variables in spherical coordinates and also by using the method of images [8], with the usual boundary conditions applied at the surface of the spherical cavity (i.e., the tangential components of the electric field are continuous; the normal components of the displacement field are continuous) [9]. Equation (12) can be interpreted as the force on the test charge arising from an approximate, induced image charge located at a distance a^2/r from the center of the spherical cavity along a common radial line with the displacement r of the test charge when r is small. The sign of the image charge is the *same* as that of the test charge, which leads to mutual *repulsion* and hence a restoring force. Consistent with the spherical symmetry of the cavity, the spring constant associated with Eq. (12) is isotropic. Hence, the center of the spherical cavity is a point of stable equilibrium for a test charge surrounded by a parelectric. This point of stability is in the vacuum. Simple harmonic motion results from small displacements in any direction, the angular frequency of which is given by

$$\Omega_{\mathrm{radial}} = \sqrt{2}\, q_{\mathrm{test}}\, m^{-1/2} a^{-3/2} \left(\frac{1-\epsilon}{1+2\epsilon} \right)^{1/2} . \qquad (13)$$

Solutions can also be obtained for configurations of several charges placed inside the spherical cavity; for instance, four identical test charges form a stable tetrahedral structure inside the cavity. Purely electrostatic traps for charged particles, which are different in principle from the Paul and the Penning traps, should be possible to construct.

An apparent contradiction arises with Earnshaw's theorem in electrostatics; i.e., no stable equilibrium exists for any static configuration of charges in a vacuum, when they are acted on by electrical forces alone. This theorem can be generalized to include conductors and dielectrics in the vicinity of these charges [10]. However, the conditions of this generalized version of Earnshaw's theorem must be incomplete, since the above solutions are counterexamples of this theorem as it is stated in [10]. This contradiction requires further study, but it should be remembered that parelectrics are metastable media which are not in thermodynamic equilibrium.

We thank Professor J. D. Jackson and Professor C. H. Townes for their comments, and Eric Bolda, Jason Bowie, Dr. John C. Garrison, Morgan Mitchell, and Aephraim Steinberg for helpful discussions, and Jeff Chamberlain and Grant McKinney for help in preparing this manuscript. J. B. is a Department of Education Graduate Research Fellow. This work was supported by the Office of Naval Research under ONR Grant No. N00014-90-J-1259.

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[1] J. P. Gordon, H. Zeiger, and C. H. Townes, Phys. Rev. **95**, 282L (1954).

[2] R. Y. Chiao, Phys. Rev. A. **48**, R34 (1993).

[3] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960).

[4] Under certain circumstances, the inverted medium will be unstable to super-radiance, to superfluorescence, or to amplified spontaneous emission. However, since in the present case the lifetime of the inverted ammonia molecules is extremely long (due to the ω^3 dependence of the Einstein A coefficient), and can be made even longer by the suppression of spontaneous emission due to placement of the system inside an off-resonance resonator, these effects are insignificant. No spontaneous emission noise is added to the transmitted wave form, since only *virtual* emissions and absorptions occur.

[5] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Oxford, 1960).

[6] R. Y. Chiao, J. Boyce, and J. C. Garrison, in Proceedings of the Conference, Fundamental Problems in Quantum Theory, in honor of Professor John Archibald Wheeler [Ann. N. Y. Acad. Sci. (to be published)]; A. M. Steinberg and R. Y. Chiao, Phys. Rev. A **49**, 2071 (1994).

[7] E. L. Bolda, R. Y. Chiao, and J. C. Garrison, Phys. Rev. A **48**, 3890 (1993); Phys. Rev. A **49**, 2938 (1994).

[8] I. V. Lindell, Am. J. Phys. **61**, 39 (1993); P. Bessemer, Am. J. Phys. **62**, 657 (1994).

[9] We solved the more general problem of a test charge placed inside a sphere of radius a filled in the interior with a dielectric ϵ_i, and filled throughout the exterior with dielectric ϵ_e. The resulting outward force on the test charge is

$$F = -\frac{q_{\mathrm{test}}^2(\epsilon_i - \epsilon_e)}{\epsilon_i a^2} \sum_{l=1}^{\infty} \frac{l(l+1)}{\epsilon_i l + \epsilon_e (l+1)} \left(\frac{r}{a} \right)^{2l-1} , \qquad (14)$$

where r is the radial displacement of the test charge from the center. The lowest order term in the special case $\epsilon_i = 1$ and $\epsilon_e = \epsilon < 1$ is given in Eq. (12). Generally there is a stable equilibrium at the center whenever $\epsilon_e < \epsilon_i$. In particular, note that a test charge immersed within a nonconducting ordinary dielectric sphere of fluid surrounded by vacuum is electrostatically stable at the center, also in apparent violation of the generalized version of Earnshaw's theorem [10].

[10] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 116ff.