# Extremal Black Holes as Exact String Solutions

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(Received 11 August 1994)

We show that the leading order solution describing an extremal electrically charged black hole in string theory is, in fact, an exact solution to all orders in $\alpha'$ when interpreted in a Kaluza-Klein fashion. This follows from the observation that it can be obtained via dimensional reduction from a five-dimensional background which is proved to be an exact string solution.

PACS numbers: 11.25.–w, 04.70.Dy

One of the goals of string theory for many years has been to find exact black hole solutions in four dimensions. The first nontrivial string ($\alpha'$) corrections to the Schwarzschild solution have been investigated [1], and solutions to the leading order equations describing charged black holes have been found [2–4]. Exact black hole solutions have been constructed in two [5] and three [6] dimensions using (gauged) Wess-Zumino-Witten (WZW) models based on the group $SL(2,R)$. In addition, using the result that the extremal limit of a black five-brane [7] is an exact superstring solution [8], one can (trivially) dimensionally reduce to obtain an exact five-dimensional extreme black hole. This black hole has a magnetic type of charge associated with the antisymmetric tensor field. It has recently been shown that one can also dimensionally reduce the exact solution in [8], down to four dimensions [9] to obtain an extreme magnetically charged black hole [10]. (These exact five- and four-dimensional black hole solutions are asymptotically flat. In addition, there are coset conformal field theory or gauged WZW constructions of just the "throat" regions of some four-dimensional extreme magnetically charged dilatonic black holes [11]. The fact that the form of the throat solution is unchanged under string corrections was noticed earlier [12].)

We shall present an exact solution describing an extremal four-dimensional *electrically* charged black hole in (super)string theory. There are several important differences between this and earlier work. First, the electrically charged solutions are qualitatively different from their magnetic analogs. The extremal magnetically charged solutions found in four and five dimensions are products of a timelike line and a Euclidean solution. So the Lorentzian nature of the solution plays no role. The extremal electrically charged solution, on the other hand, has a nontrivial timelike direction. Second, the previous four- and five-dimensional extreme black holes were shown to be exact solutions only in the superstring (or heterotic string) theory. Extended supersymmetry (on the world sheet related to that on space-time) played a key role in the argument [8] that there are no $\alpha'$ corrections to the leading order backgrounds. The solution we will discuss, like the previous two and three dimensional examples, is exact already in the bosonic string theory.

The extremal black holes we shall consider are known to be supersymmetric [13] and can be lifted to ten dimensions in such a way that they are dual to special plane fronted waves [14]. However, *a priori* these properties by themselves are not sufficient to establish that extreme black holes are solutions to all orders in $\alpha'$. We will see that another special property of the extremal electric black holes (related to a particular chiral coupling of the string to the background) can be used to show that they are exact solutions.

To satisfy the constraint on the central charge, every solution with an asymptotically flat four-dimensional space-time must have a number of internal dimensions. One could assume that the full solution is a simple product of the four-dimensional one with a small torus. However, one could equally well assume a tilted product in which off-diagonal components of the higher dimensional fields give rise to four-dimensional gauge fields. This is the approach we will adopt. We will, in fact, consider mainly the closed bosonic string theory which has no fundamental gauge fields in the higher dimensional space.

It suffices to consider one nontrivial extra dimension (with the remaining ones taken to be a trivial product). We will show that a particular five-dimensional bosonic string background is an exact solution and that its dimensional reduction yields

$$ds^2 = -F^2(r)dt^2 + dr^2 + r^2\,d\Omega\,, \quad A_t = F(r)\,,$$
$$\phi(r) = \phi_0 + \tfrac{1}{2}\ln F(r)\,, \quad F^{-1}(r) = 1 + M/r\,, \tag{1}$$

where $d\Omega$ denotes the metric of a unit 2-sphere, $A_\mu$ is the gauge field potential, $\phi$ is the dilaton, and $M$ is twice the Arnowitt-Deser-Misner mass. This background is precisely the extremal limit of the known leading order solution describing an electric dilaton black hole. In this sense one can say that the leading order extremal black hole solution is exact [15].

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Let us first describe a simple application of using dimensional reduction to find exact extremal black holes in string theory. Consider the five-dimensional plane fronted wave

$$ds^2 = du\,dv + K(r)du^2 + dr^2 + r^2 d\Omega\,, \qquad K(r) = 1 + M/r\,. \tag{2}$$

This background (with constant dilaton and no additional fields) is an exact solution to bosonic string theory [16]. To reduce to four dimensions with $u$ as the internal coordinate, we rewrite (2) in the form

$$ds^2 = K(du + \tfrac{1}{2}K^{-1}dv)^2 - \tfrac{1}{4}K^{-1}\,dv^2 + dr^2 + r^2 d\Omega\,. \tag{3}$$

Letting $t = v/2$ the four-dimensional (string frame) metric, gauge field, and scalar ($G_{uu} = e^{-2\sigma}$) are thus

$$ds^2 = -K^{-1}(r)dt^2 + dr^2 + r^2 d\Omega\,, \quad A_t = K^{-1}(r)\,, \qquad \sigma = -\tfrac{1}{2}\ln K(r)\,. \tag{4}$$

This is just the extremal electrically charged Kaluza-Klein black hole [2,10]. So the extreme Kaluza-Klein black hole can be viewed as an exact string solution.

We have recently shown [17] that a class of solutions to the leading order bosonic string equations are, in fact, exact. Included in this class was the five-dimensional fundamental string solution [18]

$$ds^2 = F(r)du\,dv + dr^2 + r^2 d\Omega\,, \tag{5}$$

$$B_{uv} = \tfrac{1}{2}F(r)\,, \quad \phi = \phi_0 + \tfrac{1}{2}\ln F(r)\,, \qquad F(r) = (1 + M/r)^{-1}\,, \tag{6}$$

where $B_{\mu\nu}$ is the antisymmetric tensor. If one writes $v = y + t$, $u = y - t$ and reduces this solution to four dimensions by assuming $y$ to be the internal coordinate, one again obtains the extreme Kaluza-Klein black hole (4) [19]. This is not surprising since the fundamental string solution (5) is related to the plane fronted wave (2) by a space-time duality transformation [20], and it turns out that the Kaluza-Klein metric is invariant under this duality. The main effect of the duality transformation is that the gauge field in four dimensions now comes from the antisymmetric tensor $B_{\mu\nu}$ and not from the off-diagonal components of the metric.

To obtain the extreme black hole (1) in four dimensions, we must start with the following generalization of the fundamental string solution:

$$ds^2 = F(r)du\,dv + du^2 + dr^2 + r^2 d\Omega\,, \tag{7}$$

with $B_{uv}$, $\phi$, and $F$ again given by (6). This background was found to be a solution of the leading order string equations in [21] and was further discussed in [22]. We will show that the arguments in [17] can in fact be extended to establish that (7) is also an exact solution. But first, we describe some properties of this solution and show that its dimensional reduction yields the extreme charged black hole.

Unlike the fundamental string (5), the metric (7) has a regular ergosphere. This can be seen by introducing new coordinates $u = y - t$, $v = 2t$, so that the metric becomes

$$ds^2 = -\left(\frac{r - M}{r + M}\right)dt^2 - \frac{2M}{r + M}\,dt\,dy + dy^2 + dr^2 + r^2 d\Omega\,. \tag{8}$$

It is now clear that the Killing vector $\partial/\partial t$, which is a unit time translation at infinity, becomes null at $r = M$. This surface is an ergosphere and not an event horizon since it is timelike. One can still travel from $r < M$ to $r > M$ provided one moves in the $y$ direction. Although the metric components remain finite at $r = 0$, the metric becomes degenerate there and the curvature diverges. One can view (7) as the extremal limit of a charged black string [7] with nonzero momentum.

If $y$ is periodically identified with a small period, the generalized fundamental string (8) will appear as four dimensional. We can obtain the effective four-dimensional geometry be rewriting (8) in the form

$$ds^2 = \left(dy - \frac{M}{r + M}\,dt\right)^2 - \frac{r^2}{(r + M)^2}\,dt^2 + dr^2 + r^2 d\Omega\,. \tag{9}$$

Thus the four-dimensional metric is precisely that of the extreme electrically charged black hole (1). Note that the dilaton is the same as in five dimensions since the "modulus" field from the dimensional reduction is constant. The gauge field now comes from both the off-diagonal components of the metric and the antisymmetric tensor which are equal (up to a gauge transformation).

We now turn to the demonstration that the background (6) and (7) [or (8)] is an exact string solution. The bosonic string in a "massless" background is described (in the conformal gauge) by the $\sigma$ model

$$I = \frac{1}{\pi\alpha'}\int d^2z\,L\,, \qquad L = (G_{\mu\nu} + B_{\mu\nu})(X)\partial X^\mu \bar{\partial} X^\nu + \alpha'\mathcal{R}\phi(X)\,, \tag{10}$$

where $G_{\mu\nu}$ is the space-time metric, and $\mathcal{R}$ is related to the world-sheet metric $\gamma$ and its scalar curvature by $\mathcal{R} \equiv \frac{1}{4}\sqrt{\gamma}R^{(2)}$. Consider a $\sigma$ model of the form

$$L_F = F(x)\partial u\,\bar{\partial} v + \partial x^i\,\bar{\partial} x^i + \alpha'\mathcal{R}\phi(x)\,. \tag{11}$$

It was shown in [17] that this model is conformally invariant to all orders (in a particular scheme) if

$$\partial^2 F^{-1} = 0\,, \quad \phi = \phi_0 + \tfrac{1}{2}\ln F(x)\,. \tag{12}$$

The $D = 5$ fundamental string solution corresponds to the case of $F^{-1} = 1 + M/r$. The key feature which enables one to establish the conformal invariance to all orders is the chiral coupling of the $u$ and $v$ fields to the background, which is obtained by setting $G_{\mu\nu} = B_{\mu\nu}$. This results in the two chiral currents associated with the null translational symmetries. It turns out that a single chiral current associated with a null symmetry is sufficient to establish the conformal invariance of similar backgrounds to all orders provided the one-loop conditions are satisfied. This means that a much larger class of leading order solutions can be shown to be exact. The most general situation will be discussed elsewhere [23]. Here we consider the following class of $\sigma$ models which includes (6) and (7) as a special case:

$$L_{FK} = F(x)\partial u \bar{\partial} v + K(x)\partial u \bar{\partial} u + \partial x^i \bar{\partial} x^i + \alpha' \mathcal{R}\phi(x). \tag{13}$$

To find the exact conditions of conformal invariance we follow [17] by introducing the source terms $L_{\text{source}} = V(z)\partial\bar{\partial}u + U(z)\partial\bar{\partial}v + X(z)\partial\bar{\partial}x$ (where $z$ denotes the two world-sheet coordinates) and performing the path integral over $v$. The resulting $\delta$ function sets $\partial u = F^{-1}\partial U$ (up to a zero mode which we absorb in $U$). We arrive at the following effective $x$ theory:

$$\begin{aligned} L'_{FK} = {} & \partial x^i \bar{\partial} x^i - F^{-1}(x)\partial U \bar{\partial} V \\ & + K(x)F^{-1}(x)\partial U \bar{\partial}\partial^{-1}[F^{-1}(x)\partial U] \\ & + \alpha'\mathcal{R}(\phi - \tfrac{1}{2}\ln F + X\partial\bar{\partial}x , \end{aligned} \tag{14}$$

where the shift in $\phi$ comes from the determinant and we make use of a special scheme to keep the free kinetic term of $x^i$ unchanged (see [17]). The conditions of conformal invariance of the $O(\partial U \bar{\partial} V)$ term are the same as for the model with $K = 0$ (12). The conformal anomaly must be local, so that only the local part of the nonlocal $O(\partial U\, \partial U)$ term may contribute to it. Since this nonlocal term already contains two factors of $\partial U$, it cannot produce $\partial X$-dependent counterterms. That means we may expand the functions $KF^{-1}$ and $F^{-1}$ in it near a constant, $x^i(z) = x_0^i + \eta^i(z)$. Then the only contractions of the quantum fields $\eta^i$ that can produce local $O(\partial U \bar{\partial} U)$ divergences are the one-loop tadpoles on the left and right sides of the nonlocal operator $\partial^{-1}$. As a result, we find the conformal invariance condition $F^{-1}\partial^2(KF^{-1}) + (KF^{-1})\partial^2 F^{-1} = 0$, or combined with (12),

$$\partial^2 F^{-1} = 0, \quad \partial^2(KF^{-1}) = 0, \quad \phi = \phi_0 + \tfrac{1}{2}\ln F(x). \tag{15}$$

When $K = 0$ these conditions obviously reduce to (12). When $F = 1$, the $\sigma$ model (13) describes the standard plane fronted wave and (15) gives the usual $\partial^2 K = 0$ condition for this background. The five-dimensional solution (7) which yields the charged black hole corresponds to the simplest nontrivial generalization (13) of the fundamental string (11) with $K = 1$.

The above discussion was in the context of the bosonic string theory. A generalization to the case of the closed superstring theory is straightforward. One only has to repeat our arguments starting with the (1,1) supersymmetric extension of the bosonic $\sigma$ model (13) (and to note that the one-loop conformal invariance conditions are the same as in the nonsupersymmetric case), with the conclusion that the above bosonic backgrounds also represent the superstring solutions.

Moreover, the same conclusion applies to the (1,0) supersymmetric heterotic extension of our $\sigma$ model, i.e., it is conformal without the need to introduce a Yang-Mills gauge field background [23]. Therefore, the exact bosonic solution (6), (7) is also an exact $D = 5$ solution of the heterotic string theory. Like the bosonic one, it can also be given a Kaluza-Klein interpretation as a four-dimensional extremal electric black hole background.

Given a $D = 4$ leading order bosonic background, its embedding into the heterotic string theory is not unique. The embeddings of extremal $D = 4$ dilatonic black holes, in which the U(1) gauge field has Kaluza-Klein and not heterotic Yang-Mills origin, have extended ($N = 2$, $D = 4$) space-time supersymmetry [13]. This suggests [24] that the corresponding world-sheet theory should presumably have (4,0) supersymmetry [25] and thus, as in the case considered in [8], the leading-order higher-dimensional superstring or heterotic string solutions should remain exact [23].

At the same time, there should also exist a related solution of the heterotic string theory formulated directly in $D = 4$. In fact, the charged dilatonic black hole may be considered as a nonsupersymmetric leading order solution of the $D = 4$ heterotic string theory, with the charge being that of the U(1) subgroup of the Yang-Mills gauge group. This solution must have an extension to higher orders in $\alpha'$ which, in general, may not be the same as the above "Kaluza-Klein" solution [26].

To summarize, we have found an exact five-dimensional string solution (6) and (7) which reduces to the extremal electrically charged black hole (1) in four dimensions. Our results further support the special nature of extremal black holes. In particular, the exactness of the leading order extremal electric black hole implies that the extreme charge to mass ratio should not be renormalized by $\alpha'$ corrections. Our method does not apply to nonextremal black holes which are likely to have $\alpha'$ corrections in all renormalization schemes. (It is easy to show that this is the case for the Schwarzschild solution.)

One of the motivations for having exact black hole solutions is to better understand the behavior of strong gravitational fields and the possible existence of singularites in string theory. Some preliminary observations are the following. In the bosonic string theory, the solution appears as four dimensional with a null singularity at low energy. However, as one approaches the singularity, one discovers that the solution is fundamentally the product of a five-dimensional solution and a torus. The five-dimensional solution has an ergosphere with a curvature singularity inside. Whether this singularity adversely affects string propagation remains to be seen. One should note that the string coupling $e^{\phi}$ goes to zero near the singularity, suggesting that quantum corrections will be suppressed.

As we noted earlier, the methods used here can be applied to a larger class of backgrounds to show that many low energy solutions are, in fact, exact. These include the recently discovered dilatonic Israel-Wilson-Perjés metrics [27,28]. The details will be given elsewhere [23].

We wish to thank G. Gibbons, R. Kallosh, and A. Strominger for useful and stimulating discussions. Part of this work was done during the LMS Durham Symposium "Quantum concepts in time and space," and we would like to express our gratitude to the organizers of this excellent meeting. G. H. was supported in part by NSF Grant No. PHY-9008502 and by EPSRC Grant No. GR/J82041. A. A. T. acknowledges the support of PPARC.

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