Critical Dynamics of Nonconserved Ising-Like Systems

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We show that the dynamical fixed point of Ising-like models, characterized by a single scalar, nonconserved ordering field, is stable near four dimensions with respect to *all* dynamic perturbations, including those of a nonequilibrium nature.

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The statistical mechanics of systems which settle into nonequilibrium steady states (NESS) has attracted considerable interest recently [1]. Motivated by their rich phenomenology, paired with simple model specifications, most effort has focused on Ising-like systems subject to a driving force: A set of Ising spins, located on the sites of a regular lattice, is updated sequentially, according to a dynamics which is local and translational invariant, in both space and time. The associated rates are controlled solely by (i) differences in internal energy, determined by a local Hamiltonian $\mathcal{H}$, (ii) the coupling to a heat bath which enters only through its temperature T, and (iii) a driving force. The latter acts as a dynamic perturbation which drives the system away from thermal equilibrium, into a steady state that is generically no longer Hamiltonian, but depends on the details of the dynamics. In its absence, the rates reduce to equilibrium form, satisfying detailed balance, so that the steady-state distribution is Hamiltonian, i.e., of Boltzmann form, *independent* of the specific choice of the dynamics.

Such nonequilibrium Ising-like systems possess many intriguing properties [1]. For example, similar to the equilibrium Ising model [2], many of these systems exhibit a continuous transition from a disordered to an ordered state [3–7]. Because of the nonequilibrium nature of the dynamics, however, several *distinct* universality classes emerge. Surprisingly, some of these are characterized by the existence of an effective, *mesoscopic* Hamiltonian, which captures the long-time long-wavelength behavior of the system. Thus, detailed balance, violated at the *microscopic* level, is restored upon coarse graining, at the fixed point of the renormalization group. For instance, a variety of two-temperature models with spin-flip dynamics [3] or combinations of spin-flip and spin-exchange dynamics [4] belong in the universality class of model A [8]. Similarly, certain two-temperature models with *spin-exchange* dynamics [5], as well as randomly driven diffusive lattice gases [6], possess an effective Hamiltonian [9]. Being long ranged, however, the latter differs from the Landau-Ginzburg-Wilson Hamiltonian which describes the equilibrium (zero-drive) system. In contrast, lattice gases driven by uniform fields violate detailed balance even at the fixed point, resulting in universal behavior distinct from any equilibrium class [7,10].

Thus, the question naturally arises whether an *a priori* criterion exists which would distinguish NESS with Hamiltonian fixed points from non-Hamiltonian ones. While a general answer is still outstanding, some progress has been made for nonequilibrium systems with *nonconserving* dynamics. Grinstein *et al.* [11] have argued that the dynamic critical behavior described by model A is stable with respect to all dynamical perturbations, including those of a nonequilibrium nature, provided (i) the dynamics is local, (ii) does not conserve the order parameter or any other auxiliary field, and (iii) respects the characteristic up-down symmetry of the equilibrium Ising model.

In this Letter, we extend this class of nonequilibrium two-state systems to include those that violate condition (iii). Thus, *all* Ising-like models with local dynamics, exhibiting continuous transitions described by a single, scalar, nonconserved field, fall into the universality class of model A. The underlying effective Hamiltonian is simply the Landau-Ginzburg-Wilson Hamiltonian of the equilibrium Ising model. More specifically, we consider the naive dimensions of all possible operators that are consistent with a continuous transition and find that there is only one coupling which, being marginal at the upper critical dimension $d_c = 4$, might destabilize the Wilson-Fisher [12] fixed point in $d < 4$. A specific microscopic realization of such a dynamics correspond to a combination of Glauber spin flips with spin exchanges biased along a preferred direction. The latter break detailed balance, as well as the up-down symmetry of the system. However, using methods of renormalized field theory [13], we show that, in $d < 4$, the potentially dangerous operator becomes irrelevant under the renormalization group (RG). Thus, the Wilson-Fisher fixed point remains stable, and the leading critical singularities are still controlled by model A. We conclude with some comments and open questions.

To investigate critical properties, it is convenient to coarse grain the microscopic dynamics, arriving at a set of mesoscopic Langevin equations for the slow variables of the theory, valid in the long wavelength long-time limit. Since we are considering a set of Ising spins undergoing a continuous transition, one of these slow variables must correspond to the local order parameter $\phi(x,t)$. For our purposes, $\phi(x,t)$ must be the *only* slow field, so that we exclude dynamical rules that would conserve other quantities such as, e.g., the energy. A general equation of motion of $\phi(x,t)$ takes the form

$$\partial_t \phi = \lambda Q[\phi, \nabla\phi] + \eta \, . \qquad (1)$$

Here, Q is an analytic function of ϕ and its derivatives, reflecting the requirement that the dynamics be local in space. Also, it contains only terms which respect the symmetries of the system. The noise term $\eta(x,t)$ models the effects of the fast microscopic degrees of freedom, after coarse graining. It has zero mean and correlations given by $\langle \eta(x,t)\eta(x',t')\rangle = N[\phi,\nabla\phi]\delta(x - x')\delta(t - t')$. The locality of the dynamics ensures that the correlations are short ranged, in *both* space *and* time, resulting in δ correlations in the long-wavelength longtime limit. $N[\phi,\nabla\phi]$ is analytic, but unrelated to Q since the dynamics may break detailed balance. Finally, $\phi(x,t)$ is not conserved, so that the right-hand side of (1) cannot be written as the gradient of a current.

We first consider the case where up-down symmetry ($\phi \to -\phi$) is present. Here, $Q[\phi,\nabla\phi]$ must be odd in ϕ, while $N[\phi,\nabla\phi]$ must be even. Model A itself corresponds to $Q_0[\phi,\nabla\phi] = -\delta\mathcal{H}/\delta\phi$, where $\mathcal{H} = \int\{\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\tau\phi^2 + \frac{1}{4!}g\phi^4\}$ takes the Landau-Ginzburg-Wilson form and Gaussian noise $N_0[\phi,\nabla\phi] = \text{const} \equiv 2\lambda$. Away from equilibrium, Q and N may of course contain additional terms, but *all* of these are irrelevant, in $d = 4 - \epsilon$, at the dynamical fixed point of model A [11]. Turning now to the case *without* up-down symmetry constraint, further operators are allowed. Fortunately, only two of these, namely

which appear in Q, are relevant on the basis of naive power counting. The former will be excluded since it induces *first order* transitions. Thus, only one new term, $\frac{1}{2}\mathbf{E}\cdot\nabla\phi^2$, must be considered.

We first note that $\frac{1}{2}\mathbf{E}\cdot\nabla\phi^2$ introduces a spatial anisotropy into the system. Without loss of generality, we choose a coordinate system such that $\mathbf{E}$ points along one of its axes, henceforth referred to as "parallel." Denoting derivatives in the parallel (transverse) subspace with ∂ (∇), our Langevin equation reads

$$\lambda^{-1}\partial_t\phi = (\nabla^2 + \rho\partial^2 - \tau)\phi - \frac{1}{3!}g\phi^3 + \frac{1}{2}E\partial\phi^2 + \eta \, , \qquad (2)$$

with $\langle\eta(x,t)\eta(x',t')\rangle = 2\sigma\delta(x - x')\delta(t - t')$. The new coupling ρ reflects the spatial anisotropy, and σ measures the strength of the noise. As usual, critically occurs for $\tau \to 0$.

To perform the perturbative analysis, we introduce a Martin-Siggia-Rose response field $\tilde{\phi}$ [14] and recast the Langevin equation as a dynamic functional [15]:

$$\mathcal{J}[\tilde{\phi},\phi] = \int d^dx\,dt\left\{\lambda\tilde{\phi}\left[\lambda^{-1}\partial_t\phi - (\nabla^2 + \rho\partial^2)\phi + \tau\phi + \frac{1}{3!}g\phi^3 - \frac{1}{2}E\partial\phi^2\right] + \sigma\tilde{\phi}^2\right\} . \qquad (3)$$

In this formalism, correlation and response functions follow as functional averages with weight $e^{-\mathcal{J}[\tilde{\phi},\phi]}$. Also, the presence (or absence) of detailed balance is easily exhibited [16]. Here, it is manifestly broken by the new operator.

Letting μ denote a characteristic momentum, naive power counting shows that g and E scale as μ^{4-d} and $\mu^{(4-d)/2}$, respectively. Thus, *both* are *marginal* operators at the upper critical dimension $d_c = 4$ and may give rise to anomalous dimensions in dimensions $d < 4$. In particular, since E is not naively irrelevant, only a renormalization group calculation can determine whether the Wilson-Fisher fixed point is stable with respect to this perturbation.

To proceed, we follow the standard methods of dynamical field theory, using dimensional regularization and minimal subtraction of the poles [13], and set up a double expansion in g and E. Since the dynamic functional (3) is invariant under a scale transformation $x_\| \to \alpha x_\|$, $\phi \to \alpha^{-1/2}\phi$, $\tilde{\phi} \to \alpha^{-1/2}\tilde{\phi}$, $\rho \to \alpha^2\rho$, $g \to \alpha g$, and $E \to \alpha^{3/2}E$, we recognize that the effective dimensionless expansion parameters of the theory are $u \equiv S_d\mu^{-\epsilon}\rho^{1/2}g$ and $v \equiv S_d\mu^{-\epsilon}\rho^{-3/2}E^2$. Here, $\epsilon \equiv 4 - d$, and $S_d \equiv 2/\Gamma(d/2)(4\pi)^{d/2}$ is a convenient geometric constant.

Of the one-particle irreducible vertex functions $\Gamma_{\tilde{N}N}$ with $\tilde{N}$ external $\tilde{\phi}$ legs and N external ϕ legs, only Γ_{11}, Γ_{20}, Γ_{12}, and Γ_{13} are primitively divergent, leading to renormalizations of τ, ρ, g, and E, as well as ϕ and $\tilde{\phi}$. Letting circles indicate bare quantities, we define the renormalization transformations via $\mathring{\phi} = \mathcal{L}_\phi^{1/2}\phi$, $\mathring{\tilde{\phi}} = \mathcal{L}_{\tilde{\phi}}^{1/2}\tilde{\phi}$, $\mathring{\lambda} = \mathcal{L}_\phi^{1/2}\mathcal{L}_{\tilde{\phi}}^{-1/2}\mathcal{L}_\lambda\lambda$, $\mathring{\tau} = \mathcal{L}_\phi^{-1}\mathcal{L}_\tau\tau$, $\mathring{\rho} = \mathcal{L}_\phi^{-1}\mathcal{L}_\rho\rho$, $\mathring{g} = \mathcal{L}_\phi^{-2}\mathcal{L}_g g\mu^\epsilon$, and $\mathring{E} = \mathcal{L}_\phi^{-3/2}\mathcal{L}_E E\mu^{\epsilon/2}$. The $\mathcal{L}$ factors are functions of u and v, minimally chosen so as to cancel the poles in the vertex functions. The Wilson β functions $\beta_u \equiv \mu\partial_\mu u|_{\text{bare}}$ and $\beta_v \equiv \mu\partial_\mu v|_{\text{bare}}$ determine the RG flow of the coupling constants u and v as the momentum scale μ varies, and thus the location and the stability of the fixed points. For our theory, they take the form

$$\beta_u(u,v) = -\epsilon u + u(\beta_u\partial_u + \beta_v\partial_v)F(u,v) \, , \qquad (4a)$$

$$\beta_v(u,v) = -\epsilon v + v(\beta_u\partial_u + \beta_v\partial_v)G(u,v) \, , \qquad (4b)$$

where $F(u,v) \equiv \frac{3}{2}\ln\mathcal{L}_\phi - \ln\mathcal{L}_g + \frac{1}{2}\ln\mathcal{L}_\rho$ and $G(u,v) \equiv \frac{3}{2}\ln\mathcal{L}_\phi - 2\ln\mathcal{L}_E + \frac{3}{2}\ln\mathcal{L}_\rho$. A simple one-loop calculation results in

$$\beta_u = -u\Big[\epsilon - \tfrac{3}{2}u - \tfrac{3}{16}v + \mathcal{O}(u^2, uv, v^2)\Big], \quad (5a)$$

$$\beta_v = -v\Big[\epsilon - \tfrac{7}{4}u - \tfrac{9}{16}v + \mathcal{O}(u^2, uv, v^2)\Big]. \quad (5b)$$

Setting $\beta_u = \beta_v = 0$, we find four fixed points, located at (a) $u^* = v^* = 0$, (b) $u^* = \frac{2}{3}\epsilon$, $v^* = 0$, (c) $u^* = 0$, $v^* = \frac{16}{9}\epsilon$, and (d) $u^* = \frac{8}{11}\epsilon$, $v^* = -\frac{16}{33}\epsilon$. Clearly, (a) is Gaussian and (b) corresponds to the Wilson-Fisher fixed point [12]. The other two fixed points, (c) and (d), which have $v^* \neq 0$, do not appear in model A.

The stability of the fixed points is determined by the eigenvalues λ_1 and λ_2 of the matrix

$$\mathbb{M}(u^*, v^*) \equiv \begin{pmatrix} \partial_u\beta_u & \partial_v\beta_u \\ \partial_u\beta_v & \partial_v\beta_v \end{pmatrix},$$

evaluated at the fixed points. For the Gaussian fixed point (a), we find $\lambda_1 = \lambda_2 = -\epsilon$. Thus, it is unstable (stable) for $d < 4$ ($d > 4$), consistent with the upper critical dimension of the theory being 4. The Wilson-Fisher fixed point (b) has eigenvalues $\lambda_1 = \epsilon$ and $\lambda_2 = \epsilon/6$, so that it is stable (unstable) for $d < 4$ ($d > 4$). Both of the new fixed points are hyperbolic, with eigenvalues ϵ and $-2\epsilon/3$ for (c) and ϵ and $-2\epsilon/11$ at fixed point (d).

The flow diagram associated with (5) is shown in Fig. 1. For our model, only the first quadrant is physical, since $v \propto E^2$, and u must be positive for stability reasons. The heavy solid lines mark the separatricies forming the boundary of the domain of attraction of the Wilson-Fisher fixed point. The two axes, $u = 0$, $v \neq 0$ and $u \neq 0$, $v = 0$, are invariant lines (to all orders in ϵ). Thus, systems with, say, $u = 0$ at some initial length scale are closed under the RG, i.e., the flow will not generate a nonvanishing u at larger length scales. The invariant line $u \neq 0$, $v = 0$ obviously corresponds to model A. The dynamics along the second invariant line, $u = 0$, $v \neq 0$, has also been studied recently [17].

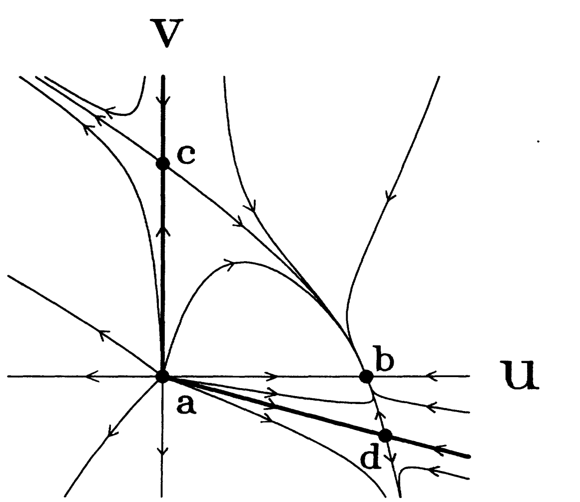

FIG. 1. Renormalization group flow in coupling constant space, generated by Eqs. (5). The arrows indicate the direction of the flow. (a)–(d) label the fixed points described in the text.

Although the present calculation does not proceed beyond the one-loop level, we expect that, at least for small ϵ, the *qualitative* topology of the flow near the origin in coupling constant space will not change at higher orders. Also, we do not anticipate that any of the *naively* irrelevant operators acquire such large anomalous dimensions as to become relevant near the Wilson-Fisher fixed point. Barring such difficulties which might affect the extrapolation of our results to physical dimensions, we arrive at the following conclusions.

For $d = 4 - \epsilon$, the stable fixed point closest to the origin is the Wilson-Fisher fixed point. Here, v and hence E vanish, so that the *fixed point* theory is just model A. Thus, the dynamic universality class of model A encompasses a wide range of driven Ising-like systems, provided their dynamics is (i) local, (ii) nonconserving, and (iii) does not involve slow fields other than a scalar order parameter. We emphasize, however, that only the fixed point, and hence the leading critical singularities, of these systems are controlled by model A. Their *subdominate* singularities are determined by irrelevant operators which are expected to differ, depending on the specifics of the microscopic rates.

Clearly, properties (i)–(iii) play an important role. Nonlocal dynamics, involving, e.g., long-ranged exchanges in addition to local spin flips [18], or simultaneous updates of whole clusters [19], exhibit dynamic critical behavior different from model A, though they may still be Hamiltonian on mesoscopic scales. Couplings to additional conserved quantities can easily generate nonmodel A-type critical dynamics, already for equilibrium theories [8]. Models involving other, nonconserved ordering fields beyond an Ising-like order parameter have (to our knowledge) not yet been fully investigated, given the large range of possible theories.

Finally, we comment briefly on comparisons with computer simulations in $d = 2$. Models satisfying the criteria established by Grinstein *et al.* [11] include spin systems with spin-flip dynamics, coupled in various ways to two heat baths at *different* temperatures. These systems unambiguously exhibit model A critical behavior [3]. To test our extension of these criteria, the microscopic dynamics must break the up-down symmetry. Wang *et al.* [20] simulated a model of this type, adding a fraction p of Glauber spin flips to the spin-exchange dynamics of the uniformly driven Ising lattice gas [3]. The drive here acts as a spatial bias, such that up (down) spins move preferentially along (against) a specific lattice direction. Such an effect clearly breaks the up-down symmetry. The induced steady-state current is modeled by an extra term $\partial\phi^2$ in the Langevin equation [10]. For $p = 0$, the dynamics is conserved, which, in conjunction with the bias, gives rise to strongly anisotropic universal behavior, characterized by anisotropic critical indices [10,1]. On the other hand, $p = 1$ correspond to the equilibrium Ising model with Glauber dynamics. Thus, for $0 < p < 1$, the model meets the premises of our analysis. The results of Wang *et al.* [20], for $p = 0.1$ and 0.5, can be summarized as follows: Isotropic exponents are observed, already at the rather small $p = 0.1$, so that the model is certainly no longer in the universality class of the (conserved) driven lattice gas. However, large crossover effects prevent a convincing measurement of Ising exponents. Instead, "effective," p-dependent indices are found which approach the Ising values. Clearly, more detailed simulations are needed.

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