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Thermodynamics of a One-Dimensional Ideal Gas with Fractional Exclusion Statistics

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We show that the particles in the Calogero-Sutherland model obey fractional exclusion statistics as defined by Haldane. We construct anyon number densities and derive the energy distribution function. We show that the partition function factorizes in the form characteristic of an ideal gas. The virial expansion is exactly computable and interestingly it is only the second virial coefficient that encodes the statistics information.

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There has been some progress recently [1,2] in understanding systems of particles with fractional exclusion statistics as defined by Haldane [3]. Motivated by physical examples, such as quasiparticles in the fractional quantum Hall systems and spinons in antiferromagnetic spin chains, Haldane formulated a generalized exclusion principle as follows: He considered many particle systems with finite dimensional Hilbert spaces where the dimension of the single particle Hilbert space depends linearly on the total number of particles present. Then the exclusion statistics parameter g is defined by $\Delta d = -g\Delta N$, where Δd is the change in the dimension of the single particle space and ΔN is the change in the number of particles. Thus g is a measure of the (partial) Pauli blocking in the system [4]. $g = 0$ (1) corresponds to bosons (fermions). In a recent work [1] we have generalized the definition of g to the case of infinite dimensional Hilbert spaces so as to apply Haldane's definition to systems in the continuum with no sharp cutoff in energy. If for such a system a virial expansion exists and all the virial coefficients are finite in the high temperature limit, then the exclusion statistics parameter g is completely determined by the second virial coefficient in this limit. In many systems it is also possible to relate g to the exchange statistics parameter α.

In a remarkable paper [2], Wu has recently considered a system with flat dispersion and, assuming that Haldane exclusion principle holds (as in the case of anyons in a magnetic field in two dimensions confined to the lowest Landau level), has shown that the statistical distribution function of this system is

$$\bar{n}(\epsilon) = \frac{1}{w(\epsilon) + g}. \tag{1}$$

Here ϵ is the single particle energy, and $w(\epsilon)$ is given as the solution of the equation

$$w^g(\epsilon, g)[1 + w(\epsilon, g)]^{1-g} = e^{\beta(\epsilon-\mu)}, \tag{2}$$

where β is the inverse temperature and μ is the chemical potential of the system. The distribution function smoothly interpolates between the Bose ($g = 0$) and Fermi ($g = 1$) distribution functions. This distribution function has also been derived earlier for the specific case of anyons confined to the lowest Landau level [5]. Wu had also analyzed the more general case of many species with different energies ϵ_i and had shown that the above distribution holds when the matrix of statistical parameters g_{ij} is of the form $g\delta_{ij}$. If the species index i could be identified with the momentum k, then indeed it could be the general form for the ideal (exclusion) anyon [6] distribution function. We will demonstrate by a first principles calculation that this is so for a model Hamiltonian system in one space dimension with inverse square law potential—the Calogero-Sutherland model (CSM) [7]. The inverse square interaction can thus be looked upon as a pure statistical interaction in one dimension.

It has lately been recognized that models with the inverse square law interaction, the CSM [7], Haldane-Shastry [8] model, and other related models, behave very much like ideal gases and that the particles have fractional statistics [9,10]. In this paper, we concentrate on the CSM, which is a model defined in the continuum. Though the spectrum of the model and its thermodynamics are exactly known [7] the ideal anyon gas nature has not been brought out so far. To this end we apply our definition of exclusion statistics [1] to it and show that the particles have fractional exclusion statistics. Next we define the anyon number density in the energy space and prove that, in the thermodynamic limit, its average is given by the distribution of the form discovered by Wu. We then show that the partition function factorizes into factors obtained by integrating the distribution function with respect to the single particle energies. This clearly demonstrates the ideal gas nature of the system. Finally, we comment on an interesting feature of the exact equation of the state.

We work in the fermionic basis of the CSM. The Hamiltonian of the system of interacting fermions in this model is given by ($\hbar = 1$)

$$H = \sum_{i=1}^{N}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\omega^2 x_i^2\right] + \frac{1}{2}\sum_{i<j=1}^{N}\frac{g(g-1)}{(x_i - x_j)^2}, \tag{3}$$

where the particles are confined in a harmonic well and the thermodynamic limit is obtained by taking $\omega \to 0$. The particles can also be put on a circle with $|x_i - x_j|$ replaced by the chord length $|\sin \pi(x_i - x_j)/L|$ in the interaction term. This model has the same thermodynamic limit as the one in Eq. (3). At special values of the coupling the model can be mapped onto particular matrix models [11].

The spectrum of CSM is exactly known. The states can be labeled by a set of fermionic occupation numbers $\{n_k\}$, $k = 1,\dots,\infty, n_k = 0,1$. The energy is given by

$$E[\{n_k\}] = \sum_{k=1}^{\infty}\epsilon_k n_k - \omega(1-g)\frac{N(N-1)}{2}, \tag{4}$$

where $\epsilon_k = k\omega$, $N = \sum_{k=1}^{\infty} n_k$. We note that this spectrum is identical to the spectrum of quasiparticle states in a Gaussian theory of compact bosons [1] with radius $R = 1/\sqrt{g}$ with the following identification: While k in the above equation is a state label in CSM, in the Gaussian theory of compact bosons k refers to the box quantized momenta. The following discussion thus applies to this case also. As can be seen from Eq. (4), the effect of the interaction is that each particle shifts the energy of every other particle by a constant $\omega(g-1)$. The energy functional can also be written as

$$E[\{n_k\}] = \sum_{k=1}^{\infty}\epsilon_k n_k - \omega(1-g)\sum_{k_1<k_2=1}^{\infty} n_{k_1}n_{k_2}. \tag{5}$$

The exact spectrum of the model is thus reproduced by an effective Hamiltonian of quasiparticle with constant density of states and constant Landau parameters [12]. As we had discussed in our earlier paper [1] this scale invariant energy shift is the basic reason for the occurrence of nontrivial exclusion statistics. We had also shown that a spectrum of the form in Eq. (4) results in the exclusion statistics parameter being equal to g.

Thus the particles in the CSM are anyons in the sense of Haldane with the exclusion statistics parameter equal to g. We will now define anyon number densities and show that their thermal average is exactly given by the distribution function in Eq. (1). To this end, we define $N(\epsilon, 0)$ as the number of particles with energy $\epsilon_k < \epsilon$, i.e., $N(\epsilon,0) = \sum_{k=1}^{\infty}\theta(\epsilon - \epsilon_k)n_k$, where $\theta(x) = 0$ for $x \le 0$ and 1 for $x > 0$. We may now define the shifted single particle energies as

$$\epsilon_A(k,g) = \epsilon_k - \omega(1-g)N(\epsilon_k, 0). \tag{6}$$

We identify this shifted energy with anyon single particle energy since the total energy can now be written as

$$E[\{n_k\}] = \sum_{k=1}^{\infty}\epsilon_A(k,g)n_k. \tag{7}$$

The number of particles with energy less than ϵ is then $N(\epsilon, g) = \sum_{k=1}^{\infty}\theta(\epsilon - \epsilon_A(k,g))n_k$. The anyon number density in the thermodynamic limit is then given by

$$n_A(\epsilon,g) = \lim_{\Delta\epsilon\to 0}\lim_{\omega\to 0}\frac{N(\epsilon+\Delta\epsilon,g) - N(\epsilon,g)}{\Delta\epsilon}, \tag{8}$$

where the limit $\omega \to 0$ is to be taken first.

We will now derive a differential equation for $\bar{N}(\epsilon,g)$, the thermal average of $N(\epsilon,g)$, and solve it to obtain the anyon distribution function. Consider a state with N particles labeled by $\{k_i\}$, $i = 1,\dots,N$ ordered such that $k_{i+1} > k_i$. Thus for this state $N(\epsilon_{k_i},0) = i-1$. The ith shifted energy is $\epsilon_A(k_i,g) = \epsilon_{k_i} - \omega(1-g)(i-1)$. Note that $\epsilon_A(k_{i+1},g) - \epsilon_A(k_i,g) \ge \omega g > 0$. Thus the anyon energies $\epsilon_A(k_i,g)$ also increase monotonically with i. We therefore have

$$N(\epsilon_{k_i},g) = N(\epsilon_{k_i},0) = i-1. \tag{9}$$

To get some feel for the anyon number densities, let us consider the ground state of the N particle system. The set $\{k_i\}$ is $(1,2,3,\dots,N)$. The corresponding set of shifted energies is $(\omega,[1+g]\omega,[1+2g]\omega,\dots,(1+[N-1]g)\omega)$. The number of particles in the interval ϵ to $\epsilon+\omega$ is then

$$n_A(\epsilon,g) = \begin{cases}\dfrac{1}{g}, & \epsilon < \epsilon_F = g\bar{\rho},\\ 0, & \epsilon \ge \epsilon_F,\end{cases} \tag{10}$$

where $\bar{\rho} \equiv \omega N$ is the average density. This is the same as the zero temperature limit of the distribution function in Eq. (1).

Next we compute how the anyon density changes with g. From Eqs. (8) and (6) we see that as g increases the particles move to the right in ϵ space with a "velocity" given by $\omega N(\epsilon,g)$. Thus the number of particles crossing a point ϵ when g increases by Δg is given by the velocity at the point of crossing multiplied by the density at that point. We thus obtain the differential equation

$$\frac{\partial \rho(\epsilon,g)}{\partial g} = \rho(\epsilon,g)\frac{\partial \rho(\epsilon,g)}{\partial \epsilon}, \tag{11}$$

where $\rho(\epsilon,g) = \omega N(\epsilon,g)$. Denoting the thermal average of $\rho(\epsilon,g)$ as $\bar{\rho}(\epsilon,g)$, and using the fact that in the thermodynamic limit $\bar{\rho}^2(\epsilon,g) = [\bar{\rho}(\epsilon,g)]^2$, we obtain the differential equation

$$\frac{\partial \bar{\rho}(\epsilon,g)}{\partial g} = \bar{\rho}(\epsilon,g)\frac{\partial \bar{\rho}(\epsilon,g)}{\partial \epsilon}. \tag{12}$$

The distribution function $\bar{n}_A(\epsilon,g)$ can be obtained from the solution of Eq. (12) by using $\bar{n}_A(\epsilon,g) = \partial\bar{\rho}(\epsilon,g)/\partial\epsilon$. Thus we need the solution to Eq. (12), with the boundary condition

$$\left.\frac{\partial \bar{\rho}(\epsilon,g)}{\partial \epsilon}\right|_{g=1} = \frac{1}{e^{\beta\epsilon}+1}. \tag{13}$$

We will now show that Eq. (12) along with the boundary condition (13) is satisfied by

$$\bar{\rho}(\epsilon,g) = \int_0^{\epsilon} d\epsilon' \frac{1}{w(\epsilon',g)+g}, \tag{14}$$

where $w(\epsilon,g)$ is determined through Eq. (2). By changing variables from ϵ to w in Eq. (14) the integral can be done to get

$$\bar{\rho}(\epsilon,g) = \frac{1}{\beta}\left[\ln\left(\frac{w(\epsilon)}{1+w(\epsilon)}\right) - \ln\left(\frac{w(0)}{1+w(0)}\right)\right]. \tag{15}$$

The condition $\bar{\rho}(\infty,g) = \bar{\rho}$ and the fact that $\lim_{\epsilon\to\infty} w(\epsilon,g) \to \infty$ implies that $w(0)$ is independent of g and $e^{-\beta\mu} = w(0)^g[1+w(0)]^{1-g}$. It can be verified that

$$\frac{\partial}{\partial g}w(\epsilon) = \frac{w+g}{w(1+w)}\bar{\rho}(\epsilon,g), \tag{16}$$

$$\frac{\partial}{\partial \epsilon}w(\epsilon) = \frac{1}{\beta}\frac{w+g}{w(1+w)}. \tag{17}$$

Using the above results it can be verified that the form of $\bar{\rho}(\epsilon,g)$ in Eq. (14) does satisfy Eq. (12). Thus the anyon distribution function in the CSM is given by

$$\bar{n}_A(\epsilon,g) = \frac{1}{w(\epsilon,g)+g}, \tag{18}$$

which is exactly the distribution function derived by Wu [2]. The additivity property of the anyon energies suggests that the grand partition function should be expressible in a factorized form. If we assume that the usual definition of the distribution function for the average number of particles holds for anyons, namely,

$$\bar{n}_A(x,g) = \frac{\partial}{\partial x}\ln z(x), \tag{19}$$

where $\ln Z^A = \int_0^\infty dx \ln z(x)$, x is the dimensionless energy defined as $x = \epsilon/\omega$. Then it follows that we must have

$$\ln Z^A = \int_0^\infty dx \ln[1 + w^{-1}(x)]. \tag{20}$$

If this were true, it would show that the particles in CSM have the basic properties of an ideal gas. We now show that this is indeed the case.

The N particle partition function for the spectrum in Eq. (4) is given by

$$Z_N^A = e^{\tilde{\beta}(1-g)N(N-1)/2}Z_N^F, \tag{21}$$

where $\tilde{\beta} = \beta\omega$ and Z_N^F is the N particle fermion partition function. Setting $g = 0$, the bosonic partition function is obtained,

$$Z_N^B = e^{\tilde{\beta}N(N-1)/2}Z_N^F. \tag{22}$$

Combining Eqs. (21) and (22) we may write the anyon partition function as

$$Z_N^A = (Z_N^F)^g(Z_N^B)^{1-g}. \tag{23}$$

The grand partition function may also be written in the form

$$Z^A = \sum_{N=0}^{\infty} e^{\beta\mu_A N}Z_N^A = \sum_{N=0}^{\infty}[e^{\beta\mu_F N}(Z_N^F)]^g[e^{\beta\mu_B N}(Z_N^B)]^{1-g}, \tag{24}$$

where we have set the anyon chemical potential $\mu_A = g\mu_F + (1-g)\mu_B$. In the thermodynamic limit, the sum is saturated at the value of $N = \bar{N}$ where $\bar{N} = \beta^{-1}\partial\ln Z^A/\partial\mu_A$. The grand partition function can then be written as

$$Z^A(\beta,\mu) = [Z^F(\beta,\mu_F)]^g[Z^B(\beta,\mu_B)]^{1-g}, \tag{25}$$

where μ_F and μ_B are determined by the conditions $\bar{N} = \beta^{-1}\partial\ln Z^F/\partial\mu_F = \beta^{-1}\partial\ln Z^B/\partial\mu_B$. The solution to these conditions is $e^{-\beta\mu_F} = w(0)$, $e^{-\beta\mu_B} = 1 + w(0)$. It can be verified that this ensures that $\beta^{-1}\partial\ln Z^A/\partial\mu_A = \bar{N}$. We therefore have

$$\ln Z^A(\beta,\mu) = g\int_0^\infty dx \ln(1 + e^{-\tilde{\beta}x-\beta\mu_F}) - (1-g)\int_0^\infty dx \ln(1 - e^{-\tilde{\beta}x-\beta\mu_B}). \tag{26}$$

We now make the change of variables $1 + e^{-\tilde{\beta}x-\beta\mu_F} = 1 + w^{-1}$ in the first term and $1 - e^{-\tilde{\beta}x-\beta\mu_B} = 1 - (1+w)^{-1}$ in the second term. We then have

$$\ln Z^A(\beta,\mu) = \int_{w(0)}^\infty dw \frac{w+g}{w(1+w)}\ln(1+w^{-1}). \tag{27}$$

We now again make a change of variable from w to ϵ using Eq. (2). The partition function then gets written as

$$\ln Z^A = \int_0^\infty dx \ln[1 + w^{-1}(x)]. \tag{28}$$

This is exactly the factorized form in Eq. (20).

We now consider the equation of state of the ideal gas of anyons in CSM. This has already been done by Sutherland [13] who, however, chose to work with the fugacity expansion. The coefficients of terms in this expansion in general depend on g and appear quite complicated. However, the coefficients of the virial expansion are extremely simple. From the product form of the partition function in Eq. (25), the virial coefficients can all be computed exactly since the virial expansion for the free Fermi and Bose gases is exactly computable when the density of states is constant. We have

$$\ln Z^A = \beta P = \bar{\rho} \sum_{l=1}^{\infty} b_l (\beta \bar{\rho})^{l-1}, \qquad (29)$$

where b_l are the virial coefficients for the ideal Fermi or Bose gases for $l \neq 2$ ($b_l^F = b_l^B \quad \forall\, l \neq 2$ for constant density of states) and $b_2 = -\frac{1}{2}(\frac{1}{2} - g)$. It is interesting to note that the interaction affects only the second virial coefficient. As we had shown [1], it is the second virial coefficient that determines the exclusion statistics. Thus the $1/r^2$ interaction in this system modifies the equation of state in a minimal way. It can therefore be thought of as a purely statistical interaction in this system.

In conclusion, we have shown by applying our definition [1] that in the CSM the particles have fractional exclusion statistics. We have then constructed the anyon number densities whose average is the ideal anyon distribution function discovered by Wu [2]. We have also shown that the partition function can be written in the factorized form characteristic of an ideal gas. Finally we have computed the virial expansion in density and have shown that only the second virial coefficient is affected by the interaction. The system thus possesses all the properties we expect of an ideal gas. Thus we have shown the system to be an ideal gas with fractional statistics.

Note added.—After submitting this paper we were informed by Yong-Shi Wu that the distribution function for the CSM has been independently derived by Bernard and Wu using the thermodynamic Bethe ansatz. We also became aware that similar results have been derived by Isakov [14].

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