

Explanation for the Resistivity Law in Quantum Hall Systems

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We consider a 2D electron system in a strong magnetic field, where the local Hall resistivity $\rho_{xy}(\mathbf{r})$ is a function of position and $\rho_{xx}(\mathbf{r})$ is small compared to ρ_{xy} . Particularly if the correlations fall off slowly with distance, or if fluctuations exist on several length scales, one finds that the macroscopic longitudinal resistivity R_{xx} is only weakly dependent on ρ_{xx} and is approximately proportional to the magnitude of fluctuations in ρ_{xy} . This may provide an explanation of the empirical law $R_{xx} \propto B dR_{xy}/dB$ where R_{xy} is the Hall resistance, and B is the magnetic field.

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Almost ten years ago, it was noticed that the longitudinal resistance R_{xx} of a quantum Hall system looks very much like the derivative of the Hall resistance R_{xy} with respect to filling fraction [1]. More recent experiments [2] have shown that, in high mobility samples, the relation

$$R_{xx} = \alpha B \frac{dR_{xy}}{dB}, \quad (1)$$

with B the magnetic field, and α a constant, holds amazingly well over a wide range of temperatures and magnetic fields, including both the integer and fractional quantized regimes, and unquantized regions where R_{xy} varies linearly with B , as for a classical system. The constant α is sample dependent, and varies somewhat with temperature, but is typically in the range 10^{-1} to 10^{-2} . Despite the simplicity and generality of this empirical relation, it has defied explanation for almost a decade. In this paper, we concentrate on temperatures that are not very low and we propose an explanation based on inhomogeneities in density over relatively long length scales.

We consider a two-dimensional electron system where the local electron density $n(\mathbf{r})$ is fixed by the charged impurity distribution in a nearby doping layer [3]. We assume that the fluctuations in the density $\delta n(\mathbf{r}) = n(\mathbf{r}) - n_0$ are much smaller than the average density n_0 , and we define a dimensionless correlation function g such that $\langle \delta n(\mathbf{r}) \delta n(\mathbf{r}') \rangle = \mu^2 g(|\mathbf{r} - \mathbf{r}'|)$ with $g(0) = 1$, and $g(x) \rightarrow 0$ as $x \rightarrow \infty$. Although we will later be concerned with the effects of disorder over several length scales simultaneously, to begin with, we will consider the case where there is only disorder on a single length scale a . As an example of a correlation function with a single length scale, we can consider $g(x) = \exp(-[x/a]^2)$. Typically, the length scale a of any disorder will be greater than or equal to the setback distance of the doping layer. If a is sufficiently long, we can define a *local* resistivity tensor $\vec{\rho}(\mathbf{r})$ which depends on the local conditions. We assume that at a fixed temperature T , the local Hall resistivity $\rho_{xy}(\mathbf{r}) = -\rho_{yx}(\mathbf{r})$ is purely a function of the local filling fraction $\nu(\mathbf{r}) \equiv \phi_0 n(\mathbf{r})/B$ with $\phi_0 = hc/e$. This assumption is certainly

valid at high temperatures, where ρ_{xy} takes on the classical value, $\rho_{xy}(\mathbf{r}) = h/\nu(\mathbf{r})e^2$. The assumption should also be valid at least as a first approximation in the ranges of T and B where the integer or fractional quantized Hall plateaus begin to develop. Then, if the density fluctuations are not too large, we may write $\rho_{xy}(\mathbf{r}) = f(\nu(\mathbf{r})) = \rho_0 + \delta\rho_{xy}(\mathbf{r})$, where $\rho_0 = f(\nu_0)$, $\nu_0 = \phi_0 n_0/B$, and $\delta\rho_{xy}(\mathbf{r}) = f'(\nu_0)\phi_0\delta n(\mathbf{r})/B$. Thus we have $\langle \delta\rho_{xy} \rangle = 0$ and $\langle \delta\rho_{xy}(\mathbf{r})\delta\rho_{xy}(\mathbf{r}') \rangle = \lambda^2 g(|\mathbf{r} - \mathbf{r}'|)$, with $\lambda = f'(\nu_0)\phi_0\mu/B$. We also assume that there is a local diagonal resistivity $\rho_{xx}(\mathbf{r})$ which may depend on the temperature and the magnetic field, but which is small compared to λ . This assumption should be quite reasonable for high mobility samples. For simplicity, we shall take $\rho_{xx}(\mathbf{r}) = \bar{\rho}$, independent of \mathbf{r} .

It should be emphasized that if T is not too low, the electronic wave function is not coherent over long length scales, due to inelastic scattering. Thus, long-range quantum interference effects are not seen. In addition, so long as the disorder is on a sufficiently long length scale, this inelastic scattering allows us to define the resistivity tensor $\vec{\rho}(\mathbf{r})$ as a local quantity. Although quantum mechanics is important in determining the value of this local tensor (through $\bar{\rho}$ and f), once the tensor is determined, we are left with a purely classical conductivity problem. We will find by analyzing this classical problem that the resistivity law (1) is relatively insensitive to the specific behavior of $\bar{\rho}$ and f . We will therefore avoid a detailed calculation of these quantities.

We now want to find the macroscopic resistivity R_{xx} for the system. The current profile $\mathbf{j}(\mathbf{r})$ is defined by Maxwell's equation $\nabla \times \mathbf{E} = 0$ and current conservation. Using $\mathbf{E} = \vec{\rho}\mathbf{j}$, we can thus define the current with the system of equations

$$\nabla \times (\vec{\rho}\mathbf{j}) = 0, \quad \nabla \cdot \mathbf{j} = 0, \quad (2)$$

along with appropriate boundary conditions and the condition that the spatial average of \mathbf{j} have a specified value \mathbf{j}_0 . By using the second equation of (2), we can rewrite the first equation as

$$(\nabla\rho_{xy} \times \mathbf{j}) + \bar{\rho}(\nabla \times \mathbf{j}) = 0, \quad (3)$$

which is clearly independent of ρ_0 . The total dissipation per unit area is given by $\frac{1}{2}R_{xx}|\mathbf{j}_0|^2$. Since the local dissipation $\frac{1}{2}\tilde{\rho}|\mathbf{j}|^2$ is independent of ρ_0 , R_{xx} must also be independent of ρ_0 . On dimensional grounds (since R_{xx} depends only on λ , $\tilde{\rho}$, and the function g), we expect that in the limit $\tilde{\rho} \rightarrow 0$ the longitudinal resistance should have a scaling form

$$R_{xx} = C\lambda^{1-\omega}\tilde{\rho}^\omega, \quad (4)$$

where C is a nonuniversal constant that depends on the form of the function g . Calculations described below suggest that the exponent ω can be close to zero, particularly if we consider the effects of disorder on several length scales simultaneously. If indeed we find that $\omega \approx 0$, then we have $R_{xx} \approx C\lambda = Cf'(v_0)\phi_0\mu/B$. On the other hand, the Hall resistance will be given by $R_{xy} = \rho_0 + \mathcal{O}(\lambda^2) \approx f(v_0)$ so $BdR_{xy}/dB = f'(v_0)\phi_0n_0/B$, thus establishing Eq. (1) with $\alpha = C\mu/n_0$.

A different model, which may be more applicable for high mobility systems at temperatures that are not too low, has dissipation arising from electron-electron scattering, which causes no dissipation for a uniform flow velocity. If the magnetic field is not too strong, we may represent this by an electron fluid viscosity η , instead of a resistivity $\tilde{\rho}$, so that the local dissipation is given by $\eta n^2|\nabla \times (\mathbf{j}/n)|^2$ rather than $\tilde{\rho}|\mathbf{j}|^2$, with \mathbf{j} the local current density. In this case we find the similar result $R_{xx} = Ca^{-2\omega}\lambda^{1-\omega}\eta^\omega$. If the exponent ω is sufficiently close to zero, so that we may neglect the field dependence of η , we may again derive the resistivity law (1).

The conductivity problem described above has been studied by various authors [4–6]. These analyses conclude that the exponent ω for the case of a local resistivity $\tilde{\rho}$, with finite range correlations in ρ_{xy} , is $\omega = \frac{3}{13} \approx 0.23$. A similar analysis for the case of a local viscosity gives $\omega = \frac{3}{19}$, as will be discussed below.

A starting point for understanding this conductivity problem is the observation that, in the limit $\tilde{\rho} \rightarrow 0$, the current $\mathbf{j}(\mathbf{r})$ can travel only perpendicular to the gradient of $\rho_{xy}(\mathbf{r})$ since the second term in Eq. (3) vanishes. Thus, for small $\tilde{\rho}$, the transport current is confined to the neighborhood of a contour of constant $\rho_{xy}(\mathbf{r})$ which percolates across the system [4,5]. If $\delta\rho_{xy}(\mathbf{r})$ is statistically symmetric about zero, then this percolating contour [7] is defined by $\rho_{xy}(\mathbf{r}) = \rho_0$. As $\tilde{\rho}$ is increased, the current spreads out somewhat from this contour. The physical interpretation of $\omega \approx 0$ is that (in either the viscous case or the resistive case) as we decrease the dissipative part of the local resistivity (fixing the total current across the system), the current profile readjusts such that the total dissipation remains approximately fixed.

If we imagine one of the percolating contours of $\rho_{xy}(\mathbf{r}) = \rho_0$ to be locally pointing in the \hat{y} direction, we can write locally $\delta\rho_{xy}(x, y) = xQ(y)$ for some function Q . Since we suspect that the current will follow along this contour, we propose that the \hat{y} component of the current

can be written in the form $\mathbf{j}_y(x, y) = JS[x/w(y)]/w(y)$, where J is the total current carried by the channel, $w(y)$ is the local width of channel, and S is an unknown function satisfying the normalization condition $\int dz S[z] = 1$. Of course, once we know \mathbf{j}_y , \mathbf{j}_x is determined by current conservation. In the small w limit it is easy to show that a solution to the system of Eqs. (2) is given for the functional form $S[z] = \exp(-z^2/2)/\sqrt{2\pi}$ when the width w of the channel is determined by the differential equation

$$d[(wQ)^p]/dy = -p\tilde{\rho}Q^{p-1}, \quad (5)$$

where $p = 2$ here. In the case of viscous dissipation, we find a differential equation similar to (5), except that $\tilde{\rho}$ is replaced by η , and $p = 4$. In the viscous case, the function S must be chosen to satisfy the differential equation $d^3S(z)/dz^3 = -zS(z)$.

As an explicit example, we consider the periodic geometry $\delta\rho_{xy}(\mathbf{r}) = \lambda\{\cos[\pi(x+y)/s] - \cos[\pi(x-y)/s]\}$ which has a square lattice of saddle points at $(x/s, y/s) \in \mathbb{Z}^2$, and we take $\tilde{\rho} \ll \lambda \ll \rho_0$. We assume an average current \mathbf{j}_0 in the \hat{y} direction so that the current flows down channels of fixed $\rho_{xy}(\mathbf{r}) = \rho_0$ along the $x = \text{integer}$ lines. Near the channel at $x = 0$, for example, we write $\delta\rho_{xy}(x, y) = xQ(y)$ with $Q(y) = -H \sin(\pi y/s)$, where H is the characteristic perpendicular slope of the function $\delta\rho_{xy}$. In this geometry we have $H = 2\pi\lambda/s$, but in a more general geometry, $H \sim \lambda/a$, where a is the characteristic length scale of the density fluctuation.

Integrating w from one saddle to the next in Eq. (5), we find that the width of the channel is given by $w(y) = \sqrt{\tilde{\rho}s/\pi H}|\sec(\pi y/2s)|$ in the resistive case. Of course the small w approximation breaks down near the points where this expression diverges, so the solution must take a different form very close to these points. The important thing to extract from this result is that the characteristic width W of the channel scales as $W \sim \sqrt{\tilde{\rho}s/H}$. Numerical solutions [8] of the system of Eq. (2) to find the exact current profile in this periodic geometry (as well as in other simple geometries) support this conclusion. Since we have determined the current profile, we can also calculate the local dissipation $\tilde{\rho}|\mathbf{j}|^2$ and thus extract the resistance of this current-carrying channel. In general, we find that the resistance of a channel of length s scales as $R_s \sim \tilde{\rho}s/W \sim HW$. A similar argument for the case of viscous dissipation yields the results $W \sim (\eta s/H)^{1/4}$ and $R_s \sim \eta s/W^3 \sim HW$.

We must remember, however, that we do not actually have a regular array of saddle points. We have instead a tortuous conduction network. We define a percolation problem by choosing a cutoff ϵ , considering all points such that $|\delta\rho_{xy}(\mathbf{r})|/\lambda < \epsilon$ to be “conducting” points, and all other points to be “insulating.” This type of problem [7] has a critical value of $\epsilon_c = 0$, provided we take $\delta\rho_{xy}$ to be statistically symmetric about zero. Our actual system will have thin channels of width W centered at the percolating $\delta\rho_{xy} = 0$ lines, and we will require for

self-consistency that $\epsilon \sim W/a$ for these channels. The percolation network has a weblike topology consisting of essentially one-dimensional conduction paths that snake through the system and intersect each other once every distance ξ . More properly, the correlation length ξ is defined as the size of the largest insulating islands in the conducting network. As we approach the percolation threshold by letting $\epsilon \rightarrow 0$, the conduction channels become thinner, connections between conduction paths are broken, and the correlation length diverges as $\xi \sim a\epsilon^{-\nu}$. For the case where the disorder has only one length scale, it is known that $\nu = \frac{4}{3}$ [4,7,9].

Although the conduction path in this percolation problem can branch somewhat [4,7,9] on scales smaller than ξ , forming dead-end loops and small multiply connected regions, it is assumed for now that the current effectively follows a one-dimensional path around the perimeter of these branched regions [4,6]. The length of such a path connecting points separated by a distance ξ scales as $a(\xi/a)^D$, where D is the fractal dimension of the path. For this particular problem, where there is only one length scale, it is believed that $D = \frac{7}{4}$ [4].

Because of the weblike topology of the network, we can think of the percolation network as an effectively periodic system similar to the one we analyzed above [4,6]. Here the periodicity length is ξ . However, since the channels are not straight, the length s of the conduction channels must be taken to be the typical path length $a(\xi/a)^D$ between intersections of macroscopic conduction paths. Thus we have $W \sim \sqrt{\bar{\rho}s/H} \sim a\sqrt{\bar{\rho}\xi^D/\lambda a^D}$. On the other hand, our self-consistency condition requires $W \sim a\epsilon$. Using $\xi \sim a\epsilon^{-\nu}$, we find $\bar{\rho} \sim \lambda\epsilon^{2+\nu D}$. Since the system is homogeneous on length scales larger than ξ , and since there is typically only one conduction path per square of length ξ on a side, the macroscopic longitudinal resistivity is given by $R_{xx} \sim R_s \sim HW \sim \lambda\epsilon \sim \lambda^{1-\omega}\bar{\rho}^\omega$ where $\omega = (2 + \nu D)^{-1} = \frac{3}{13}$ in the case of a single disorder length scale. This result agrees with the previous work of Refs. [4] and [6]. An analogous calculation can be performed for the viscous case with a single disorder length scale (using $W \sim [\eta s/H]^{1/4}$ instead of $W \sim \sqrt{\bar{\rho}s/H}$) to give the result $\omega = (4 + \nu D)^{-1} = \frac{3}{19}$.

Although these arguments seem quite reasonable and are in decent agreement with existing numerical work [4], it is possible that they are not exact. In particular, we are not very certain of the fractal dimension D of the conduction path. For example, if the current were to readjust itself only very slightly so as to cut off dead-end loops, D might be changed. However, it is clear that $1 \leq D \leq 2$, and this uncertainty allows only a small range of values of ω ($\frac{3}{14} \leq \omega \leq \frac{3}{10}$ in the resistive case, and $\frac{3}{20} \leq \omega \leq \frac{3}{16}$ in the viscous case).

We can also show rigorously, for a slightly modified model, that the exponent ω can never be negative. In the modified model, we take $\rho_{xy}(\mathbf{r}) = \rho_0 e^{h(\mathbf{r})}$ and $\rho_{xx}(\mathbf{r}) = \bar{\rho} e^{h(\mathbf{r})}$, where $h(\mathbf{r})$ is dimensionless here and

has a symmetric distribution about zero. This model is statistically self-dual, in the sense that for any member A of the ensemble with a local resistivity tensor $\vec{\rho}_A(\mathbf{r})$, there is another B , having equal probability, with $[\vec{\rho}_B(\mathbf{r})]^\dagger = [\vec{\rho}_A(\mathbf{r})]^{-1}(\rho_0^2 + \bar{\rho}^2)$. It can be shown that the ensemble-averaged resistivity tensor \vec{R} satisfies $\vec{R}^\dagger = \vec{R}^{-1}[\rho_0^2 + \bar{\rho}^2]$, which implies $R_{xx}^2 + R_{xy}^2 = \rho_0^2 + \bar{\rho}^2$. [The proof employs a transformation analogous to that used by Dykhne and Ruzin [10], with the choice $a = c = 0$, $d = b^{-1} = (\rho^2 + \bar{\rho}^2)^{1/2}$ in Eq. (8) of Ref. [10].] Therefore, R_{xx} cannot diverge for $\bar{\rho} \rightarrow 0$, and we must have $\omega \geq 0$. More generally, one expects that the exponent ω will always be ≥ 0 for any model where the macroscopic resistivity is isotropic [4].

We now consider the effect of having disorder on multiple length scales. As an extreme example, let us consider a system that is disordered on a microscopic length scale a (as above) and also disordered on a macroscopic length scale a' such that $a \ll a' \ll L$, where L is the size of the system. On scales much less than a' , the calculation described above should hold; i.e., given a local resistivity tensor $\vec{\rho}$ such that ρ_{xy} has some typical fluctuation λ , and $\rho_{xx} = \bar{\rho}$, we can use the above approach to calculate that the resistivity tensor $\vec{\rho}'$ measured on a scale larger than ξ but smaller than a' is given by $\rho'_{xx} = \bar{\rho}' \sim \lambda^{10/13} \bar{\rho}^{3/13}$ with ρ'_{xy} given by the local mean value of ρ_{xy} . We can then repeat the argument to account for disorder on the scale a' by using the tensor $\vec{\rho}'$ as a local resistivity. Here, we similarly find that the macroscopic resistivity is given by $R_{xx} \sim \lambda^{10/13} (\bar{\rho}')^{3/13} \sim \lambda^{160/169} \bar{\rho}^{9/169}$. Thus the exponent ω is effectively squared if disorder exists on two very different length scales. Clearly, this approach can be extended analogously to account for disorder on any number of well-separated length scales, with the result that the exponent ω can become arbitrarily small if the disorder extends over a sufficiently wide range of scales—i.e., if the correlations fall off sufficiently slowly. Isichenko has previously considered the effect of long-range correlated disorder and has similarly found that ω can be arbitrarily small for certain correlations that fall off very slowly [4].

The hypothesis that disorder exists on several length scales seems plausible. Although one certainly expects to find disorder on the scale of the dopant setback distance, there is also evidence for inhomogeneities in the electron density on the scale of a few percent [11], and the length scale of these inhomogeneities may be relatively large. In this regard, it is interesting that the values of σ_{xx} obtained from the surface acoustic wave experiments of Willett *et al.* using a conventional theoretical analysis, in the low frequency limit, are systematically smaller by a factor of ≈ 3 than the values obtained from macroscopic conductivity measurements on similar samples, over a range of fields and temperatures. (This discrepancy was removed by an explicit renormalization of a parameter σ_m in the experimental analysis [11,12].) If the sample has

sufficiently large density fluctuations on a length scale large compared to the ultrasound wavelength together with smaller fluctuations on a shorter length scale, this could conceivably account for the observed discrepancy. However, there remains a problem in accounting for the experimental values of the coefficient α in Eq. (1) via large scale inhomogeneities. Analysis by Dykhne and Ruzin [10] in the case where $\rho_{xy}(\mathbf{r})$ can only take on two discrete values, ρ_1 and ρ_2 , gives a maximum value of R_{xx} equal to $|\rho_1 - \rho_2|/2$, when the value of $\tilde{\rho}$ is small. Assuming this estimate to be an upper bound on R_{xx} for the continuum case, we see that in order to obtain a coefficient $\alpha \approx 0.03$, as in typical experiments [2], one would need to have large scale density fluctuations of order 6%, which is probably unreasonably large.

If the exponent ω were precisely zero, then the resistivity law (1) would hold independent of the microscopic details of the local dissipative resistivity (or viscosity) so long as the local dissipation is sufficiently small. If ω is small, but nonzero, then the accuracy of the law will be determined to some extent by the behavior of the local resistivity. In the high temperature regime, for example, if the local resistivity $\tilde{\rho}$ itself varies as B^α with $\alpha \approx \frac{1}{2}$, then at high temperatures the macroscopic resistivity would vary as $B^{1-\omega/2}$.

Theoretical results that in various particular circumstances, within the quantum Hall regime, the macroscopic longitudinal resistivity is determined by local fluctuations of the electron density $n(\mathbf{r})$, or equivalently of the local Hall resistance $\rho_{xy}(\mathbf{r})$, have been discussed previously by a number of authors [6,10,12,13]. Much of this previous work has concentrated on the narrow transition regions between neighboring well-developed Hall plateaus at low temperatures. In the present Letter, we have been more concerned with the regime of high temperature, where $\rho_{xy}(\mathbf{r})$ may be more properly considered to be a continuous function of the local filling factor and hence a continuous function of position in the sample. Moreover, we have shown that in the high temperature regime a percolation analysis of the conductivity problem may be used to explain the empirical resistivity law (1) which should hold particularly well if disorder occurs over a wide range of length scales. Further analysis of the inhomogeneities occurring in actual samples is clearly necessary to complete this argument, as is a better understanding of dis-

sipation processes at the shortest length scales, which determine $\tilde{\rho}$.

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