Adiabatic Quantum Transport: Quantization and Fluctuations

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Quillen's local index theorem is used to study the charge transport coefficients (adiabatic curvature) associated with the ground state of the Schrödinger operator for charged (spin less) particles on a closed, multiply connected surface. The formula splits the adiabatic curvature into an explicit integral part and a fluctuating part depending on the regularized determinant of the Hamiltonian.

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Some of the interesting recent developments in quantum transport are connected with two complementary phenomena: the precise quantization of certain transport coefficients (e.g., conductance) [1] and the fluctuation of others [2]. Normally the two are viewed as disparate phenomena that have little to do with each other: They pertain to different kinds of physical systems and to different notions of conductances, and the theoretical frameworks used to describe the two seem to have little in common.

Here we shall study a class of quantum systems for which the transport coefficients simultaneously display quantization and fluctuation in a way that, we believe, sheds light on both. A precise description of the kinds of systems we consider shall be given below. For the moment, however, these may be thought of as roughly describing the dynamics of a Schrödinger particle on a two-dimensional multiply connected surface which has no boundary but is of *finite area*. We consider the nondissipative transport in a constant magnetic field and $2h$ Aharonov-Bohm fluxes. Each flux penetrates one of the fundamental loops of the surface.

There are two related notions of transport coefficients that we consider: *conductance* and *charge transport*. Changing one of the fluxes generates electromotive force (emf) around the corresponding loop and as a result of that current flows around possibly all other fluxes. In the limit of small emf's, currents and emf's are related by the antisymmetric conductance matrix. We define the matrix of charge transport so that each entry in the matrix is an integral over a pair of fluxes of the corresponding element in the conductance matrix [formula (12)].

The tool we bring to bear is a local index theorem due to Quillen [3]. When applicable, it splits the conductances into two parts [formula (14)]. The first is explicit and universal, i.e., it is—up to a factor—the canonical symplectic form on the space of Aharonov-Bohm fluxes. Its two-dimensional integrals yield quantized charge transport (in units of e^2/h) and therefore provide a connection to the integral quantum Hall effect [1,4]. The second piece in the formula is an exact 2-form, hence it does not affect charge transport. It affects, however, the conductance as a fluctuation term. In contrast to the first one, it depends on spectral properties of the Hamiltonian, since it is related to an appropriate regularization of its determinant.

In systems which are classically chaotic one expects a relation between the eigenvalues and the lengths of closed classical orbits. This relation is exact in situations where Selberg's trace formula applies [5] and is a semiclassical approximation in cases where Gutzwiller's formula holds [6]. In some special cases (see below) the determinant of the Hamiltonian—and hence the second piece of formula (14)—can be expressed in terms of Selberg's zeta function.

There are several applications to this point of view; all of them are consequences of formula (14) below: (1) All transport coefficients associated with the ground state for any compact Riemann surface with h handles and sufficiently large magnetic field B are explicitly calculated. They are ± 1 if the pair of fluxes is associated with intersecting loops and 0 otherwise. This solves a problem posed in [7], also solved, by different means, in [8]. (2) Conductance fluctuations in the ground state, in the limit of large magnetic fields, are exponentially small. (3) In special cases where we can apply Selberg's trace formula, conductance fluctuations in the ground state are related to the periodic orbits of the corresponding classical system.

Now we shall describe the model in more detail. Consider a charged, nonrelativistic, spinless particle on a topologically closed surface Σ with h handles. Fix a complex structure on it with local coordinate z and a conformal metric $\rho(z,\bar z)|dz|^2$. The area element is the 2-form $\sigma = \frac{i}{2}\rho\, dz \wedge d\bar z$.

The magnetic field is a real 2-form $B\sigma$. We allow only constant magnetic fields (relative to σ) with total magnetic flux $2\pi f = \int_\Sigma B\sigma$, where f is an integer.

For B fixed consider the space $\mathcal{A}$ of real gauge potentials. For any $A \in \mathcal{A}$ we have $dA = B\sigma$, where locally on Σ

$$A = \bar A^{0,1}(z,\bar z)\, dz + A^{0,1}(z,\bar z)\, d\bar z\,. \qquad (1)$$

To show explicitly the connection between gauge potentials and Aharonov-Bohm fluxes we introduce on Σ a canonical basis of 1 cycles, i.e., simple oriented loops $\gamma_1, \dots, \gamma_{2h}$, which consist of h disjoint pairs γ_j, γ_{j+h} (one pair for each handle) such that γ_j intersects γ_{j+h} exactly at one point. Consider the dual basis $a_1, \dots, a_{2h}$ of real harmonic 1-forms on Σ normalized by

$$\int_{\gamma_j} a_k = \delta_{jk}, \qquad j,k = 1,\dots,2h. \tag{2}$$

Fix some $A_0 \in \mathcal{A}$; then any gauge potential $A \in \mathcal{A}$ is uniquely represented in the form

$$A = A_0 + \sum_{j=1}^{2h} \phi^j a_j + i\, dg\, g^{-1}, \tag{3}$$

where $g : \Sigma \to U(1)$ is a gauge transformation. Factorizing the space $\mathcal{A}$ by the action of the gauge group, we obtain a $2h$-dimensional torus Φ parametrized by the Aharonov-Bohm fluxes.

The flux torus Φ carries natural symplectic and complex structures. (Both of them are necessary to formulate Quillen's theorem.) In terms of the Aharonov-Bohm fluxes the symplectic structure is given by

$$\Omega = \sum_{i=1}^{h} d\phi^j \wedge d\phi^{j+h}. \tag{4}$$

To define the complex structure, consider the Hodge star operator on real harmonic 1-forms on Σ,

$$*(\alpha\, dz + \bar{\alpha}\, d\bar{z}) = i(\bar{\alpha}\, d\bar{z} - \alpha\, dz). \tag{5}$$

Since $*^2 = -1$, we can regard $*$ as a multiplication by i; this gives rise to a complex structure on Φ which we denote by J. It is explicitly given by $J = \Omega\, g$, where g is the Riemannian metric on Φ,

$$g_{ik} = \int_\Sigma a_i \wedge *a_k. \tag{6}$$

Now we return to the discussion of particles on Σ in the presence of a constant magnetic field B. The family of Schrödinger operators we consider is defined through the kinetic energy of a spinless particle on Σ including a gauge potential A and is given by

$$H(\phi) = (d + iA)^*(d + iA) = 4D^*D + B, \tag{7}$$

where $D = \frac{\partial}{\partial \bar{z}} + iA^{0,1}$ and $D^* = -\frac{1}{\rho}(\frac{\partial}{\partial z} + i\bar{A}^{0,1})$. The Schrödinger operator acts on the Hilbert space H of sections of a U(1) line bundle with the scalar product

$$(\psi, \psi) = \int_\Sigma |\psi|^2 \sigma. \tag{8}$$

Let us recall a few basic facts about adiabatic transport [1,4]. Let $P(\phi)$ denote the spectral projection on the ground state of $H(\phi)$ [9]. Suppose that the fluxes $\phi^j(t)$ depend adiabatically on time. Then, the current around the kth flux is given by

$$I_k(P, \phi) = i \sum_{j=1}^{2h} \dot{\phi}^j \omega_{jk}(P, \phi). \tag{9}$$

Hence $\omega_{jk}(P, \phi)$ is, up to a factor i, the conductance matrix. In mathematical terms it is a component of the adiabatic curvature 2-form $\omega\,(P, \phi)$ [10]:

$$\omega(P, \phi) = \mathrm{Tr}(P\mathbf{d}P \wedge \mathbf{d}P) = \mathrm{Tr}[P(\mathbf{d}H)\widehat{(H - E)}^{-2}(\mathbf{d}H)], \tag{10}$$

where $\mathbf{d}$ denotes exterior derivative with respect to flux and $\mathbf{d}P = \sum(\partial P / \partial \phi^i)\mathbf{d}\phi^i$. The hat in $\widehat{(H - E)}^{-1}$ excises the pole at E (the reduced resolvent). The expression on the right-hand side of Eq. (10) is Kubo's formula in operator notation. In the case at hand, the adiabatic curvature is also given by the formula [3]

$$\omega(P, \phi) = \mathrm{Tr}\left(P\mathbf{d}D^* \frac{1}{DD^*}\mathbf{d}D \right). \tag{11}$$

The integral of $\omega_{jk}(P, \phi)$ over the fluxes ϕ^j and ϕ^k has a direct physical interpretation: It is the total charge transported around loop k (averaged over flux ϕ^k) as a result of the adiabatic increase of flux through loop j by one quantum unit. This defines the charge transport matrix Q_{jk} [11]:

$$Q_{jk} = \frac{i}{2\pi} \int_{0 \le \phi^j, \phi^k \le 2\pi} \omega_{jk}(P, \phi)\, d\phi^j \wedge d\phi^k. \tag{12}$$

In the situation we consider, Quillen's local index theorem gives an explicit formula for the curvature form of the so-called determinant bundle of the family of Cauchy-Riemann operators D, parametrized by Aharonov-Bohm fluxes ϕ. To apply Quillen's formula to the analysis of adiabatic curvature, notice first that by Riemann-Roch

$$\dim \ker D - \dim \ker D^* = f - h + 1. \tag{13}$$

By a standard positivity argument $\ker D^* = 0$ when $f \ge 2h - 1$. Then $\dim \ker D = f - h + 1$ is constant on the flux space, and Quillen's theorem yields the following formula for conductance:

$$\frac{i}{2\pi} \omega(P, \phi) = -\frac{1}{(2\pi)^2}\Omega - \frac{1}{4\pi}\, dJ\, d \ln \det D^*D. \tag{14}$$

Here $\det D^*D$ is defined via the zeta function regularization. The determinant term is an exact 2-form on Φ, hence it does not contribute to the charge transport matrix Q_{jk}, and represents conductance fluctuations. The natural symplectic form Ω on the flux torus depends only on the topology of the surface and in this sense is universal. Furthermore, it determines charge transport. Thus, Quillen's theorem divides conductance into a constant and a fluctuating term.

In addition to the consequences of formula (14) mentioned at the beginning, we can draw the following conclusions: (1) Formula (14) suggests that the behavior of conductance is qualitatively more complicated for small magnetic fields, i.e., $f = 0, 1, \ldots, 2h - 2$. (2) It is known that for the special case of the flat two torus $\det D^*D$ is constant on the flux torus. Hence conductance has no fluctuating term. This relates well to the fact that classical dynamics on the flat torus is integrable. (3) More interesting is the case of surfaces of constant negative curvature -1 and integral magnetic field B with total flux $f = (2h - 2)B$. These surfaces are conveniently realized as the orbit spaces of discrete subgroups Γ of $\mathrm{SL}(2, R)$ acting on the upper half plane H. In this case the determinant of the Hamiltonian can be expressed in terms of Selberg's zeta function,

$$Z(s, \phi) = \prod_{\text{primitive}\,\gamma} \prod_{k=0}^{\infty} [1 - \chi(\gamma, \phi) e^{-(s+k)\ell(\gamma)}] . \quad (15)$$

Here $\gamma \in \Gamma$ are primitive hyperbolic elements of Γ representing conjugacy classes and may be thought of as closed geodesics on Σ; $\ell(\gamma)$ denotes the length of γ and $\chi(\gamma, \phi)$ are phase factors that carry information about the fluxes:

$$\chi(\gamma, \phi) = \exp i \left(\sum_{1}^{2h} \phi_j n_j(\gamma) \right), \quad (16)$$

where $n_j(\gamma)$ counts the number of times (including signs) the closed geodesic γ goes around the jth fundamental loop. Here we choose A_0 in formula (3) to be the canonical connection of the Hermitian, holomorphic line bundle $(T^*\Sigma)^{\otimes B}$ [i.e., compatible with both the metric and the holomorphic structure in $(T^*\Sigma)^{\otimes B}$]. The series on the right-hand side of (15) converges absolutely for $\mathrm{Re}\, s > 1$ and has an analytic continuation with respect to s to the whole complex plane. The determinant of D^*D is expressed in terms of Selberg's zeta function by [12]

$$\det D^*D = c_h Z(B, \chi), \qquad B \geq 2, \quad (17)$$

where c_h is a constant independent of the fluxes. When B becomes large, $Z(B, \chi)$ tends to one and its derivatives decrease exponentially. Therefore conductance fluctuations are exponentially small for large magnetic fields. Moreover, the leading term in the asymptotics of the fluctuating term for large B is given by $\exp\{-[B\ell(\gamma_{\min})]\}$, where $\gamma_{\min}$ is the shortest homologically nontrivial closed geodesic.

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