

## Crossover from High to Low Reynolds Number Turbulence

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The Taylor-Reynolds and Reynolds number ( $Re_\lambda$  and  $Re$ ) dependence of the dimensionless energy dissipation rate  $c_\epsilon = \epsilon L / u_{1,rms}^3$  is derived for statistically stationary isotropic turbulence, employing the results of a variable range mean field theory. Here  $\epsilon$  is the energy dissipation rate,  $L$  the (fixed) outer length scale, and  $u_{1,rms}$  a rms velocity component. Results for  $c_\epsilon(Re_\lambda)$  and also for  $Re_\lambda(Re)$  are in good agreement with experiment. Using the  $Re$  dependence of  $c_\epsilon$  we account for the time dependence of the mean vorticity  $\omega(t)$  for decaying isotropic turbulence. The lifetime of decaying turbulence, depending on the initial  $Re_{\lambda,0}$ , is predicted to saturate at  $0.18L^2/\nu \propto Re_{\lambda,0}^2$  ( $\nu$  the viscosity) for large  $Re_{\lambda,0}$ .

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Dimensional analysis has proven to be a powerful tool in turbulence research, giving a number of key features of turbulent spectra [1,2]. The main idea is to connect small scale quantities such as the energy dissipation rate  $\epsilon$  with the large scale quantities such as the outer length scale  $L$  and a rms velocity component,  $u_{1,rms}$ . More precisely, following Richardson's cascade picture of turbulence [2,3], it is argued that for fully developed turbulence

$$\epsilon = c_\epsilon u_{1,rms}^3 / L \quad (1)$$

holds, where  $c_\epsilon$  is a dimensionless constant in the range of 1.

However, it is not clear, for what Reynolds number  $Re$  the large  $Re$  limit (1) is reached. For small  $Re$  the dimensionless number  $c_\epsilon$  clearly depends on  $Re$ . For example, for plane Couette flow with shear  $2u_{1,rms}/L$  in one direction we have  $c_\epsilon(Re) = 4/Re$ . Here and henceforth, following Sreenivasan's [4] collection of measurements for grid turbulence, we defined the Reynolds number  $Re$  as

$$Re = Lu_{1,rms}/\nu, \quad (2)$$

where  $\nu$  is the viscosity and  $L$  is the longitudinal integral scale as in [4]. Even for large  $Re$  it is not clear whether the  $Re$  dependence of  $c_\epsilon$  vanishes, though Ref. [4] favors  $c_\epsilon = \text{const}$  for high  $Re$ .

Another way of expressing the physical contents of (1) is to give the  $Re$  dependence of the Taylor-Reynolds number  $Re_\lambda = \lambda u_{1,rms} / \nu$ , where  $\lambda = u_{1,rms} / (\partial_1 u_1)_{rms}$  is the Taylor microscale. With  $\epsilon = 15\nu(\partial_1 u_1)_{rms}^2$ , valid for isotropic turbulence [2], and Eqs. (1) and (2) we get

$$Re_\lambda = \sqrt{15Re/c_\epsilon(Re)}. \quad (3)$$

If we have  $c_\epsilon = \text{const}$  for large  $Re$ , we thus also have  $Re_\lambda \propto \sqrt{Re}$ .

In this Letter we will first derive *explicit* expressions for  $c_\epsilon(Re)$  and  $c_\epsilon(Re_\lambda)$  for unbounded flow. They depend only on the Kolmogorov constant  $b$ , which is experimentally *known* to be between 6 and 9 [5-7]. We will start from the results of the variable range mean field theory [7], which embodies the Navier-Stokes *dynamics*, yet neglecting intermittency

effects [8]. We thus offer a way to go far beyond dimensional analysis. In the second part of the Letter we apply our results to decaying turbulence, which has recently been experimentally examined and analyzed by dimensional analysis by Smith, Connelly, Goldenfeld, and Vinen [9]. Both our results for  $c_\epsilon(Re_\lambda)$  and for the time dependence of the mean vorticity  $\omega(t)$  in decaying turbulence are in good agreement with experimental data.

We start from the final result of Effinger and Grossmann's variable range mean field theory [7]. For homogeneous, isotropic turbulence a differential equation for the velocity structure function  $D(r) = \langle [u(x+r) - u(x)]^2 \rangle$  is derived [7], namely

$$\epsilon = \frac{3}{2} \left( \nu + \frac{D^2(r)}{b^3 \epsilon} \right) \frac{1}{r} \frac{d}{dr} D(r). \quad (4)$$

For small  $r$  the solution is  $D(r) = \epsilon r^2 / 3\nu$ , whereas for large  $r$  we get the well-known Kolmogorov result  $D(r) = b(\epsilon r)^{2/3}$ . The Kolmogorov constant  $b$  can be calculated within the approach of Ref. [7];  $b = 6.3$  is obtained. Equation (4) is an energy balance equation. The first term in parentheses corresponds to viscous dissipation; the second one can be interpreted as eddy viscosity. Equation (4) constitutes some kind of closure of the Navier-Stokes dynamics. Other kinds of closure are possible and will lead to quantitatively only slightly different results.

Integrating the differential equation (4) from 0 to the outer length scale  $L$  we get

$$D^3(L) + 3b^3 \epsilon \nu D(L) - b^3 \epsilon^2 L^2 = 0. \quad (5)$$

Here we have assumed that (4) holds for all  $r$  up to  $r = L$ . If  $L$  is large enough and the flow is isotropic, it is  $D(L) = 2\langle u^2 \rangle = 6u_{1,rms}^2$ . With Eqs. (1) and (2) the quadratic (in  $\epsilon$ ) Eq. (5) can be written in dimensionless form as  $c_\epsilon^2 - 18c_\epsilon/Re - (6/b)^3 = 0$ , which is easily solved to give

$$c_\epsilon(Re) = c_{\epsilon,\infty} \left\{ \left( \frac{3b^3}{8} \right)^{1/2} \frac{1}{Re} + \sqrt{1 + \frac{3b^3}{8} \frac{1}{Re^2}} \right\}. \quad (6)$$

The dependence of  $c_\epsilon$  on the Taylor-Reynolds number  $Re_\lambda$  can be obtained from Eqs. (3) and (6),

$$c_\epsilon(Re_\lambda) = c_{\epsilon,\infty} \sqrt{1 + \frac{5}{4} \frac{b^3}{Re_\lambda^2}}. \quad (7)$$

In both formulas we have introduced  $c_{\epsilon,\infty} = (6/b)^{3/2}$  only for *convenience*; the  $Re$  and  $Re_\lambda$  dependences are purely determined by the Kolmogorov constant  $b$ .

For large  $Re$  or  $Re_\lambda$ , the function  $c_\epsilon(Re_\lambda)$  indeed becomes *constant*,  $c_\epsilon(Re_\lambda) = c_{\epsilon,\infty} = (6/b)^{3/2}$ , as Sreenivasan finds for grid turbulence with biplane square mesh grids [4]. The experimental value  $c_{\epsilon,\infty} \approx 1.0$  is only at the borderline of the numerical value  $c_{\epsilon,\infty} \approx 0.54-1.0$  (for  $b = 9-6$ ) of our prediction. The reason for this is likely due to nonuniversal boundary effects. We have assumed (4) to hold up for all  $r$  up to  $r = L$ ; thus  $D(L) = 6u_{1,rms}^2 = b(\epsilon L)^{2/3}$ , which already results in  $c_{\epsilon,\infty} = (6/b)^{3/2}$ . Yet for  $r$  around  $L$ ,  $D(r)$  will not scale as  $D(r) = b(\epsilon r)^{2/3}$  and we have  $D(L) < b(\epsilon L)^{2/3}$ . This nonuniversal boundary effect might be treated by introducing an effective geometry  $b^{(eff)} < b$  instead of  $b$  in Eq. (5), defined by  $D(L) = b^{(eff)}(\epsilon L)^{2/3}$ . [Note that an introduction of  $b^{(eff)}$  already in (4) is inappropriate, as boundary effects should not be seen in  $D(r)$  for small  $r$ .] To get the experimental  $c_{\epsilon,\infty} = 1.0$ , one should have  $b^{(eff)} = 6/c_{\epsilon,\infty}^{2/3} = 6 \leq b$ .

The curve  $c_\epsilon(Re_\lambda)$  is plotted in Fig. 1, together with Sreenivasan's experimental data for grid turbulence. For  $Re_\lambda \approx 50$  the function  $c_\epsilon(Re_\lambda)$  saturates at  $c_{\epsilon,\infty}$ , in good agreement with the data.

For really small  $Re$  Eq. (6) can—strictly speaking—no longer be applied, as laminar flow is never isotropic, whereas Eq. (4) holds only for turbulent, isotropic flow. Note that Fig. 1, starting with  $Re_\lambda = 5$ , does not include the laminar case (as seen from the inset), since in laminar

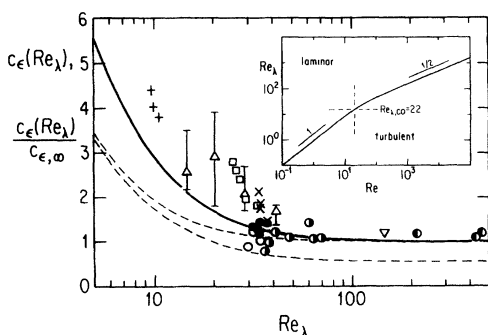


FIG. 1.  $c_\epsilon(Re_\lambda)$  for  $b = 6$  (upper dashed) and for  $b = 9$  (lower dashed) according to Eq. (7), compared with Sreenivasan's collection of experimental data [4]. The solid line shows the normalized curve  $c_\epsilon(Re_\lambda)/c_{\epsilon,\infty}$  (for  $b = 8.4$ ), which can be compared with the same experimental data points as the experimental  $c_{\epsilon,\infty}$  equals 1 [4]. The normalization leads to better agreement with the data for small  $Re_\lambda$ . The inset shows  $Re_\lambda(Re)$  according to Eqs. (3) and (6). The crossover from the laminar scaling  $Re_\lambda \propto Re$  to the turbulent range scaling  $Re_\lambda \propto \sqrt{Re}$  takes place at  $Re_{\lambda,co} = 22$ .

flow  $Re_\lambda$  loses its meaning. If we perform the small  $Re$  limit  $Re \ll \sqrt{3b^3/8} = 14.9$ ,  $Re_\lambda \ll \sqrt{5b^3/4} = 27.2$ ; nevertheless, we can get  $c_\epsilon(Re) = 18/Re$ , independent of  $b$ , as expected, since  $b$  characterizes the highly turbulent state. (Here and in what follows, we took  $b = 8.4$ , which is given in [7] as the experimental value.) The  $\propto Re^{-1}$  dependence is correct. The prefactor 18 is—again, as expected—too large, if compared to the highly anisotropic laminar Couette flow with shear in only one direction, see above. In more isotropic laminar flow, e.g., in flow with shear in *three* directions, the agreement for small  $Re$  will be better.

Equations (3) and (6) give the function  $Re_\lambda(Re)$ , see inset of Fig. 1, which strongly resembles the latest experimental measurements for grid turbulence by Castaing, Gagne, and Marchand [10]. Also Grossmann and Lohse obtain a similar curve  $Re_\lambda(Re)$  from a reduced wave vector set approximation of the Navier-Stokes equations [11]. Here, for large  $Re$  we have  $Re_\lambda = \sqrt{15(b/6)^{3/2}} \sqrt{Re} \approx 5\sqrt{Re}$ . For small  $Re$  it is  $Re_\lambda = \sqrt{5/6} Re$ . Extrapolating these two limiting cases, the crossover between them takes place at  $Re_{co} = 18(b/6)^{3/2} = 29.8$ , corresponding [via Eq. (3)] to  $Re_{\lambda,co} = 21.6$ , which seems very realistic to us.

Up to now we applied our theory to statistically stationary turbulence. But it also offers an opportunity to analyze decaying turbulence. Most experiments on decaying turbulence have been performed in wind tunnels up to now, where the distance from the grid gives the decay time  $t$ , if the mean velocity is known [12]. In this kind of experiment the outer length scale  $L$  grows with time  $t$ , as the wake behind the grid becomes wider with increasing distance. Yet in a very recent new type of experiment performed by Smith *et al.* [9],  $L$  can be kept fixed. In that experiment a towed grid generates homogeneous turbulence in a channel filled with helium II. The decay of the mean vorticity  $\omega(t)$  is measured by second sound attenuation [9,13]. As in [9] we assume that Navier-Stokes dynamics can be applied to this fluid.

In the theoretical analysis of this experiment, for very high  $Re_0 = Re(t = 0)$ ,  $Re_{\lambda,0} = Re_\lambda(t = 0)$ , the quantity  $c_\epsilon$  can be considered constant. But the smaller  $Re(t)$  or  $Re_\lambda(t)$  become with increasing time  $t$ , the more important are the corrections seen in Eqs. (6) and (7). The total energy (per unit mass) of the flow is  $E = u_{rms}^2/2 = 3u_{1,rms}^2/2$ . The decay of the fully developed turbulence is governed by the differential equation

$$\dot{\epsilon} = c_\epsilon \frac{u_{1,rms}^3}{L} = c_\epsilon [E(t)] \left(\frac{2}{3}\right)^{3/2} \frac{E^{3/2}(t)}{L} = -\dot{E}(t). \quad (8)$$

The outer length scale  $L$  is fixed, as in the experiment we are referring to [9]. We change variables to  $Re(t) = \sqrt{2/3} L \sqrt{E(t)}/\nu$  and obtain

$$\dot{Re} = -\frac{1}{3} \frac{\nu}{L^2} c_\epsilon(Re) Re^2. \quad (9)$$

Integrating Eq. (9) with the initial condition  $\text{Re}(t=0) = \text{Re}_0$  gives the time dependence  $\text{Re}(t)$  of the Reynolds number,

$$\frac{t}{\tau} = \frac{3}{c_{\epsilon,\infty}} \int_{\text{Re}_0}^{\text{Re}(t)} \frac{-dx}{\gamma x + x\sqrt{x^2 + \gamma^2}}. \quad (10)$$

Here, for simplicity, we have introduced the viscous time scale  $\tau = L^2/\nu$  and the constant  $\gamma = \sqrt{3b^3/8} = 9/c_{\epsilon,\infty} = 14.9$ . The integral can be calculated analytically. We define the indefinite integral as

$$F(\text{Re}) = \frac{1}{2\text{Re}^2} \left\{ -\gamma + \sqrt{\gamma^2 + \text{Re}^2} \right\} + \frac{1}{2\gamma} \ln \left\{ \frac{\gamma + \sqrt{\gamma^2 + \text{Re}^2}}{\text{Re}} \right\}. \quad (11)$$

Thus the time dependence of  $\text{Re}(t)$  is given by the inverse function of

$$t(\text{Re})/\tau = 3[F(\text{Re}) - F(\text{Re}_0)]/c_{\epsilon,\infty}. \quad (12)$$

All quantities on the right-hand side can be expressed in terms of the Kolmogorov constant  $b$ .

Imagine now the limiting case of large  $\text{Re}_0$  and also large time, but  $\text{Re}(t) < \text{Re}_0$  still large, i.e.,  $t$  not too large. For large  $\text{Re}$  both terms in (11) contribute  $(2\text{Re})^{-1}[1 + O(\gamma/\text{Re})]$ . Thus  $F(\text{Re}) = 1/\text{Re}$  and from (12) we explicitly get

$$\text{Re}(t) = \left( \frac{1}{\text{Re}_0} + \frac{c_{\epsilon,\infty}}{3} \frac{t}{\tau} \right)^{-1} \approx \frac{3}{c_{\epsilon,\infty}} \left( \frac{t}{\tau} \right)^{-1}. \quad (13)$$

In Fig. 2 we plotted  $\text{Re}_\lambda(t)$ , calculated from Eqs. (3) and (12), for several  $\text{Re}_0$ . The scaling law  $\text{Re}_\lambda(t) \approx 3\sqrt{5}(t/\tau)^{-1}/c_{\epsilon,\infty}$  [corresponding to (13)] starts to be observable only for  $\text{Re}_0 \approx 10^3$ , i.e.,  $\text{Re}_{\lambda,0} \approx 156$ .

In the final period of decay, i.e., for very large  $t$  [large enough so that  $\text{Re}(t) \ll \gamma$ ], we get  $\text{Re}(t) = 2\gamma\sqrt{e} \times \exp(-6t/\tau)$  and  $E(t) = (9/4)eb^3\nu^2L^{-2} \exp(-12t/\tau)$ . An exponential decay for very large  $t$  also holds for decaying turbulence with growing outer length scale  $L(t)$  [12].

To compare our results with the helium II experiment [9], we have to calculate the mean vorticity  $\omega(t)$ . Vorticity always causes strain in the flow. It can be shown [2] that  $\nu\omega^2 = \epsilon$ . Thus  $\nu\omega^2 = \epsilon = -\dot{E} = -3\nu^2\text{Re}\dot{\text{Re}}/L^2$ . With Eq. (9) we get

$$\begin{aligned} \tau\omega(t) &= \text{Re}\sqrt{\text{Re}c_\epsilon(\text{Re})} \\ &= \sqrt{c_{\epsilon,\infty}} \text{Re}(t) \sqrt{\gamma + \sqrt{\gamma^2 + \text{Re}^2(t)}}, \end{aligned} \quad (14)$$

where the universal law on the right-hand side depends only on the Kolmogorov constant  $b$  and on the time  $t$ .

Next, we estimate the lifetime  $t_l$  of the decaying turbulence. For this purpose we calculate how the Kolmogorov length [2]  $\eta(t) = [\nu^3/\epsilon(t)]^{1/4} = \sqrt{\nu/\omega(t)}$  depends on time  $t$ . Of course,  $\eta(t)$  will increase with time, as the turbulence becomes weaker and weaker. The behavior can be obtained from Eqs. (14) and (12) for any  $\text{Re}_0$ . If  $\text{Re}_0$  is large enough, scaling  $\eta(t)/L \propto (t/\tau)^{3/4}$  can develop.

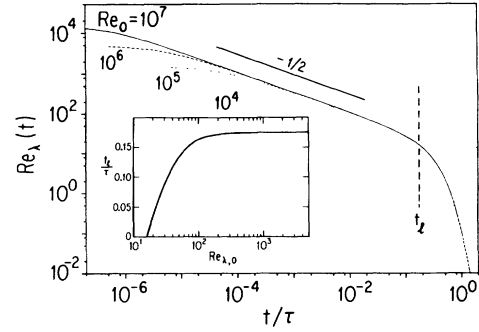


FIG. 2. Time dependence of the Taylor-Reynolds number  $\text{Re}_\lambda(t)$  for freely decaying turbulence with fixed outer length scale for different  $\text{Re}_0 = 10^7, 10^6, 10^5, 10^4$  ( $\text{Re}_{\lambda,0} = 1.58 \times 10^4, 4.98 \times 10^3, 1.58 \times 10^3, 498$ ), left to right, cf. Eqs. (3), (11), and (12) with  $b = 8.4$ . The dashed line denotes the lifetime  $t_l$ , i.e., the time when viscous effects start to dominate, see text. The inset shows the  $\text{Re}_{\lambda,0}$  dependence of the lifetime  $t_l$  of the turbulence, calculated from Eqs. (15), (11), and (3) with  $b = 8.4$ .

How do we define the lifetime  $t_l$  of the turbulence? As the crossover in the structure function  $D(r)$  between the viscous subrange and the inertial subrange happens at  $r \approx 10\eta$  [7], we define the lifetime  $t_l$  by the condition  $10\eta(t_l) = L$ . A Reynolds number  $\text{Re}_l$  is associated with this time  $t_l$  via Eqs. (14) and the definition of  $\eta$ . We calculate  $\text{Re}_l = 20.3$ ,  $\text{Re}_{\lambda,l} = 16.0$ , which is, as it should be, near to the viscous-turbulent crossover in the curve  $\text{Re}_\lambda(\text{Re})$ , which occurs at  $\text{Re}_{c0} = 29.8$ ,  $\text{Re}_{\lambda,c0} = 21.6$ ; see inset of Fig. 1. With this definition we obtain the lifetime  $t_l$  of the decaying turbulence for any given  $\text{Re}_0$  [or, via (3),  $\text{Re}_{\lambda,0}$ ] as

$$t_l(\text{Re}_0)/\tau = 3[F(\text{Re}_l) - F(\text{Re}_0)]/c_{\epsilon,\infty}. \quad (15)$$

We plotted  $t_l(\text{Re}_{\lambda,0})$  in the inset of Fig. 2. For small  $\text{Re}_{\lambda,0}$  the lifetime  $t_l(\text{Re}_{\lambda,0})$  grows logarithmically with  $\text{Re}_{\lambda,0}$ . For very large  $\text{Re}_{\lambda,0}$  it saturates at  $t_l(\text{Re}_{\lambda,0}) = 3\tau F(\text{Re}_l)/c_{\epsilon,\infty} = 0.18\tau$ ; i.e., it becomes independent on  $\text{Re}_0$ , if measured in time units of  $\tau = L^2/\nu$ . The lifetime, if measured in seconds, of course increases  $\propto \text{Re}_0 \propto \text{Re}_{\lambda,0}^2$  in the limiting case.

Finally we compare with the data of the helium II experiment [9]. First we have to embody the boundary effects, as they should be larger in the helium II experiment than in grid turbulence, because the turbulence decays in a tube. Indeed, in Ref. [9]  $c_{\epsilon,\infty}^{(\text{eff})} = 36.4$  is given [using our definition of  $c_\epsilon$ , Eq. (1)], corresponding to  $b^{(\text{eff})} = 6/(c_{\epsilon,\infty}^{(\text{eff})})^{2/3} = 0.55$ .

Using this  $b^{(\text{eff})}$  instead of  $b$  (or  $\gamma^{(\text{eff})}$ ,  $c_{\epsilon,\infty}^{(\text{eff})}$ , respectively) in Eqs. (11) and (14), we plotted  $\omega(t)$  in Fig. 3, together with Smith *et al.*'s experimental data [9]. The two curves show the same features. For small  $t$  there is no power law. For medium  $t$  the theory gives  $\tau\omega(t) \approx 3^{3/2}(t/\tau)^{-3/2}/c_{\epsilon,\infty}^{(\text{eff})}$ . The power law exponent

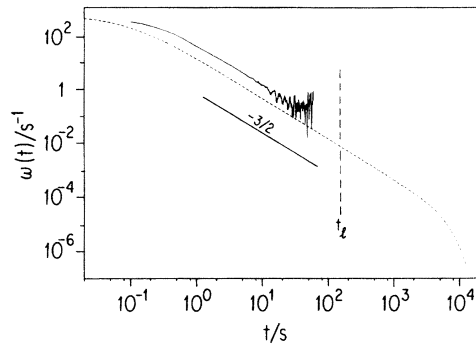


FIG. 3. Time dependence of the mean vorticity  $\omega(t)$  from the experiment of Smith, Donnelly, Goldenfeld, and Vinen [9] for  $\text{Re}_{\lambda,0} = 634$ , solid line. In that experiment  $L = 1$  cm,  $\nu = 8.97 \times 10^{-5}$  cm<sup>2</sup>/s, so that  $\tau = L^2/\nu = 1.11 \times 10^4$  s. The dashed line is our prediction for  $\omega(t)$  according to Eq. (14) with  $b^{(\text{eff})} = 0.55$ . The corresponding lifetime  $t_l = 0.013\tau = 140$  s is given, too.

$-3/2$ , which is clearly seen in the experimental data, has already been derived by dimensional analysis [9]. But in theory the  $-3/2$  power law for  $\omega(t)$  extends much further than observable in experiment, because the experimental noise hinders observation already for  $t > 10$  s. For further comparison a reduction of the experimental noise is essential. Experiments with dramatically increased sensitivity of the detectors are in progress [14].

The theoretical lifetime  $t_l$  of the decaying turbulence (calculated with  $b^{(\text{eff})} = 0.55$ ) is  $t_l = 0.013\tau = 140$  s, much larger than the 10 s in which the  $\omega(t)$  signal can be measured. Thus viscous effects, arising from the Re dependence of  $c_\epsilon$  for smaller Re, see Eq. (6), only become important for a time  $t_l \gg 10$  s. So the slight decrease in the measured  $\omega(t)$  signal for  $t_l \approx 10$  s is not due to them, as one might have thought, but possible due to the uncoupling of the normal and superfluid components of helium II (which was used as the fluid in the experiment [9]), as speculated in [9].

We summarize our main results. We first calculated the functions  $c_\epsilon(\text{Re}_\lambda)$  and  $\text{Re}_\lambda(\text{Re})$ , Eqs. (3), (6), and (7), from a variable range mean field theory [7], which goes far beyond dimensional analysis. We then applied our results to decaying turbulence, highlighted by the expression for the time dependence of  $\omega(t)$ , Eq. (14). All results are in good agreement with experiment. To even improve the agreement, future work has to be done to embody nonuniversal boundary effects in this approach. A way to do so is to introduce  $b^{(\text{eff})}$  instead of  $b$  in Eq. (5).

Alternatively, one could get rid of the boundary effects by calculating a high passed filtered velocity field from the experimental data, so that the nonuniversal effects are filtered out.

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