Amplification under the Standard Quantum Noise Limit

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We show that a saturable amplifier can beat the "3 dB" quantum noise figure of linear amplifiers, approaching the ideal "0 dB" Heisenberg limit. Noise suppression is due to nonlinear saturation effects which cut off the high-intensity tail of the photon distribution. A Monte Carlo simulation of an actual semiconductor laser amplifier leads to a minimum noise figure of 0.51 dB, with a still sizable gain of 10.8 dB. The best performance corresponds to an output signal photon number which equals 0.7 times the saturation number.

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The noise of quantum origin in optical amplifiers poses severe limitations to the transparency of optical networks [1], particularly at the first amplification stages of the communication line, but also at repeaters and boosters which split the line into several branches. The most significant characteristics of the amplifier are the gain $\mathcal{G}$ and the noise figure $\mathcal{R}$. These are defined as follows:

$$\mathcal{G} = \frac{S_{\text{out}}}{S_{\text{in}}}, \qquad \mathcal{R} = \frac{(S^2/\mathcal{N})_{\text{in}}}{(S^2/\mathcal{N})_{\text{out}}}, \tag{1}$$

where S and $\mathcal{N}$ denote signal and noise at the input (in) and at the output (out) of the amplifier. For on-off modulation one has

$$\begin{aligned} S &= \langle \hat{O} \rangle_{\text{on}} - \langle \hat{O} \rangle_{\text{off}}, \\ \mathcal{N} &= \frac{1}{2} \left(\langle \Delta \hat{O}^2 \rangle_{\text{on}} + \langle \Delta \hat{O}^2 \rangle_{\text{off}} \right), \end{aligned} \tag{2}$$

where $\langle \cdots \rangle$ denotes ensemble average and $\hat{O}$ is the detected observable. Hence, both gain and noise generally depend on the kind of detection performed at the output of the amplifier.

Optimal performance is achieved when the amplifier operates at the required gain with the lowest possible noise figure. The Heisenberg uncertainty relations limit the minimum achievable noise figure of any device to $\mathcal{R} \geq 1$ ($\geq$0 dB) (otherwise it would be possible to devise a detection apparatus which is able to measure an observable classically, upon reducing its quantum fluctuations as desired [2]). The ideal Heisenberg limit can actually be approached for homodyne detection using parametric amplification (one quadrature of the field is amplified, whereas the conjugated quadrature is attenuated, both with ideal noise figure; see Refs. [3] and [4]). Therefore phase sensitive parametric amplifiers are ideal amplifiers for homodyne detection.

In the following we will focus attention only on direct detection, as it is still an open problem to design an optimal number amplifier working at the Heisenberg limit. In the quasimonochromatic approximation, the detected observable is given by

$$\hat{O} = h\nu \Delta\nu a^\dagger a, \tag{3}$$

where $a^\dagger a$ is the number operator of the field mode a at the peak frequency ν, and $\Delta\nu$ is the optical bandwidth of the amplifier. For direct detection ideal photon number amplifiers have been suggested in Ref. [5], and ideal Hamiltonians have been derived in Ref. [6] but with no feasible device approaching the ideal behavior. (For parametric conversion in some special circumstances it is possible to attain quasi-ideal amplification but only for very low gains [4]. Recently, a scheme for realizing a photon number amplifier has also been suggested, based on a high-quantum-efficiency photodetectors and a number state semiconductor laser [7].)

In the above scenario the minimum $\mathcal{R} = 2$ ($\simeq$3 dB) noise figure for linear amplifiers has been consolidated as an actual standard limit for direct detection, and it is generally considered as the unavoidable noise which is due to spontaneous emission (or parametric spontaneous emission for parametric amplifiers [4]). Such a limit, however, is just a consequence of the linearity assumption. In fact, let us briefly recover the standard 3 dB limit from the Fokker-Planck equation of a linear amplifier. In the Glauber-Sudarshan P-function representation the field evolution is given by

$$\begin{aligned} \frac{\partial}{\partial t} P(\alpha, \alpha^*, t) = &-\frac{1}{2}(A - C) \\ &\times \left[\frac{\partial}{\partial \alpha} [\alpha P(\alpha, \alpha^*, t)] + \text{c.c.} \right] \\ &+ A \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, \alpha^*, t), \end{aligned} \tag{4}$$

where α is the complex radiation field, A is the stimulated emission coefficient, and $C = B + 2k$ is the sum of the absorption coefficient and the cavity loss. The gain and the noise figure for direct detection can be easily evaluated from normal ordered moments. They are given by

$$\mathcal{G} = \exp[(A - C)t], \tag{5}$$

$$\begin{aligned} \mathcal{R} = 1 + \frac{1}{\mathcal{G}^2 \Delta n_0^2} \{ &n_0 \mathcal{G}(\mathcal{G} - 1)(2D - 1) \\ &+ 2(\mathcal{G} - 1)D[(\mathcal{G} - 1)D + 1] \}. \end{aligned} \tag{6}$$

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In Eq. (6) $D = A/(A - C)$, whereas n_0 and Δn_0^2 represent the average and fluctuations of the input photon number, respectively. For high gains ($A \gg C$) and intense coherent inputs, the noise figure approaches the value

$$\mathcal{R} \simeq 1 + \frac{n_0}{\Delta n_0^2} = 2. \tag{7}$$

It is of great interest to establish whether the limit (7) is effectively a lower bound for real devices or not. It has been pointed out that saturation effects may significantly reduce the output noise associated with spontaneous emission [8,9], but the obvious objection is that they also reduce the gain dramatically. A first evidence of very low noise figures for significant gains has been obtained in Ref. [10], where an actual semiconductor laser amplifier has been modeled according to the Scully-Lamb theory (see Ref. [11]). However, this result is affected by unphysical noise figures under the Heisenberg limit (due to insufficient consideration of the amplified spontaneous emission), and hence the problem has still remained open.

In order to clarify the role of saturation effects in reducing the output noise we adopt a very accurate Fokker-Planck equation developed by Lugiato, Casagrande, and Pizzuto [12]. In the Wigner function representation $W(x, x^*, t)$, the equation is given by

$$k^{-1}\frac{\partial W(x,x^*,t)}{\partial t} = \left\{\frac{\partial}{\partial x}\left[x\left(1 - \sigma\frac{2\Theta}{1+|x|^2}\right)\right] + \text{c.c.} - \frac{\Theta}{2N_s}\frac{\partial^2}{\partial x^2}x^2\frac{(1+|x|^2)^2 + \sigma^2(1+2f)}{(1+|x|^2)^3} + \text{c.c.}\right.$$
$$\left. + \frac{1}{N_s}\frac{\partial^2}{\partial x \partial x^*}\left[1 + \Theta\frac{(1+|x|^2)^2(2+|x|^2) - \sigma^2(1+2f)|x|^2}{(1+|x|^2)^3}\right]\right\}W(x,x^*,t). \tag{8}$$

In Eq. (8) k is the cavity loss, N_s is the saturation proton number, x is the scaled complex field ($x = \alpha/\sqrt{N_s}$, α being the eigenvalue of the annihilation operator), σ is the atomic population inversion, Θ is the cooperation parameter, and $f = \gamma_\parallel/2\gamma_\perp$ ($\gamma_\parallel$ and $\gamma_\perp$ are the longitudinal and transverse atomic decay rates, respectively). Essentially, all approximations made in deriving Eq. (8) from the microscopic master equation rely on the assumption that $N_s \simeq N \gg 1$ [13], with N denoting the effective number of atoms populated on the two lasing levels. Moreover, because of adiabatic elimination of atomic variables, the following inequality should also be satisfied [14]:

$$\Theta > \frac{1}{4(1-\sigma)}. \tag{9}$$

The advantage of using the Wigner-function representation (instead of the P-function) is that the diffusion matrix always remains positive definite for any choice of parameters and all values of the field [thus also $W(\alpha, \alpha^*, t)$ remains positive at all t for input coherent states at $t = 0$].

It is easy to check that the Fokker-Planck equation (8) in the P-function representation correctly reduces to the linear case (4) in the limit of weak signals and large saturation numbers N_s. In this way one also obtains the following relations among parameters:

$$A = 2k\Theta(1+\sigma), \qquad B = 2k\Theta(1-\sigma). \tag{10}$$

In the nonlinear regime Eq. (8) can be efficiently solved by means of Green's-function Monte Carlo techniques [15]. Here we focus attention on the same semiconductor amplifier considered in Ref. [10], whose characteristic parameters are reported in Table I. We consider input coherent states and evaluate gain and noise figure according to Eqs. (1)–(3). The mean photon number and the number fluctuations are related to the Wigner-function moments as follows:

$$\langle \hat{n}(t)\rangle = \int d^2\alpha\, W(\alpha,\alpha^*,t)|\alpha|^2 - \frac{1}{2}, \tag{11}$$

$$\langle \Delta\hat{n}^2(t)\rangle = \int d^2\alpha\, W(\alpha,\alpha^*,t)(|\alpha|^4 - |\alpha|^2) - \langle \hat{n}(t)\rangle^2. \tag{12}$$

In our Monte Carlo approach the simulated points are distributed in the complex plane according to the distribution $W(\alpha, \alpha^*, t)$. Input coherent states correspond to Gaussian Wigner functions. After evolving points according to Eq. (8), the mean values (11) and (12) are evaluated simply as sums over simulated data. The error bars are computed, as usual, by dividing the ensemble of points into subensembles, and then calculating the rms deviation of the subensemble averages with respect to the global one.

TABLE I. Characteristic parameters of the semiconductor (GaAs) laser amplifier analyzed in this paper (from Ref. [10]). Parameters k, Θ, and σ are obtained through Eq. (10) in the radiative limit $f = 1$.

Stimulated emission coefficient	A	1.17×10^{12} s^{-1}
Absorption coefficient	B	3.375×10^{11} s^{-1}
Cavity damping constant	k	1.875×10^{11} s^{-1}
Speed of light	v	7.5×10^7 m s^{-1}
Cavity damping length	l_*	1.2×10^3 μm
Peak frequency	ν	1.96×10^{14} s^{-1}
Optical bandwidth	$\Delta\nu$	1.263×10^{13} s^{-1}
Cooperation parameter	Θ	2.01
Saturation photon number	N_s	1.115×10^4
Population inversion	σ	0.5522
Parallel/transverse decay ratio	f	1
Photons per input power	$(h\nu\Delta\nu)^{-1}$	6.1×10^5 W^{-1}

In Fig. 1 we plot the gain versus the amplifier length for different input signals. Three qualitatively different operating regimes are evident: (i) a *linear* regime for small signals ($x \ll 1$ for all $l \leq l_*$), (ii) a *saturating* regime leading to $x < 1$ at $l \sim l_*$, and (iii) an *ultrasaturating* regime where $x \sim 1$ at $l \sim l_*$. Qualitatively, in the linear regime the gain increases almost linearly with l for the whole amplifier length; in the saturating regime the gain starts saturating roughly at the middle of the amplifier length, whereas in the ultrasaturating regime it is already completely saturated and starts decreasing. In Fig. 2 the noise figure is plotted for the same input signals of Fig. 1. The effect of saturation is striking. In the linear regime the noise figure starts linearly, then approaches an asymptotic value as a function of the amplifier length. In the saturating regime the noise figure is no longer monotonically increasing and exhibits a deep minimum before $l = l_*$. Finally, in the ultrasaturating regime the noise figure exhibits a plateau region at approximately 3 dB before the complete gain saturation, where it again starts increasing as a function of l.

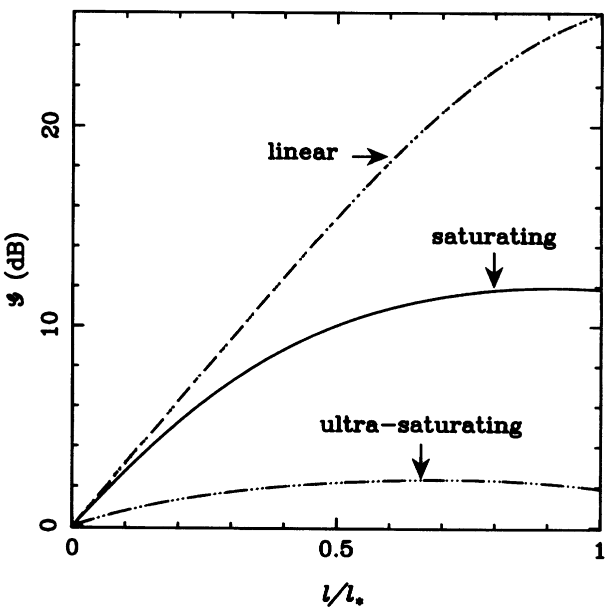

FIG. 1. Gain versus amplifier length for the semiconductor laser amplifier in Ref. [10] (the parameters of the amplifier are reported in Table I). The amplifier has been modeled according to the Fokker-Planck equation (8) using 20 Monte Carlo experiments of 1000 events each (the fixed spontaneous amplification emission has been more carefully studied with 200 experiments for the input vacuum). The characteristics of the amplifier have been evaluated for input coherent states, direct detection, and on-off modulation according to Eqs. (1), (2), (11), and (12). The three different regimes correspond to (i) *linear* for $S_{\rm in} = -20$ dB m; (ii) *saturating* for $S_{\rm in} = 0.16$ dB m; and (iii) *ultrasaturating* for $S_{\rm in} = 10.72$ dB m [S(dB m) $= 10 \log_{10}(10^3 S)$: the equivalent average photon number is $\langle n \rangle = S/h\nu\Delta\nu$].

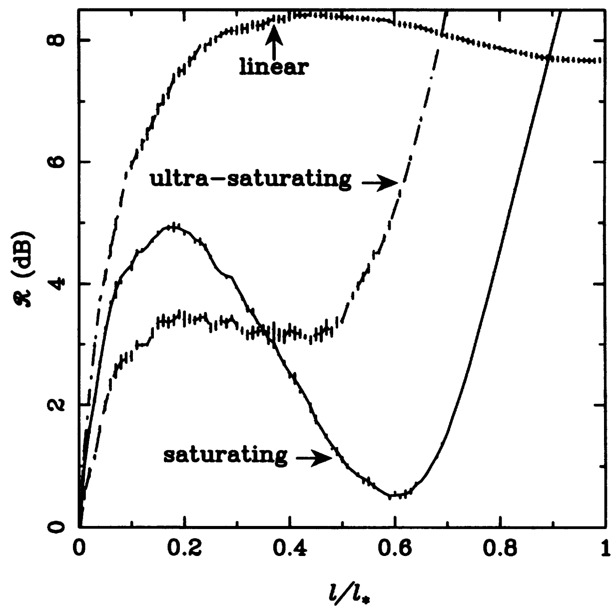

FIG. 2. Noise figure versus amplifier length for the same parameters of Fig. 1.

In Figs. 3 and 4, the gain and noise figure are plotted as levels curves versus both the input signal and the amplifier length. A wide region where the noise figure is well below the standard 3 dB limit is evident, corresponding to still sizable gains. The minimum is $\mathcal{R} = 0.51$ dB, corresponding to a gain $\mathcal{G} = 10.8$ dB and an amplifier length $l = 0.6 l_*$. It occurs for the output signal photon

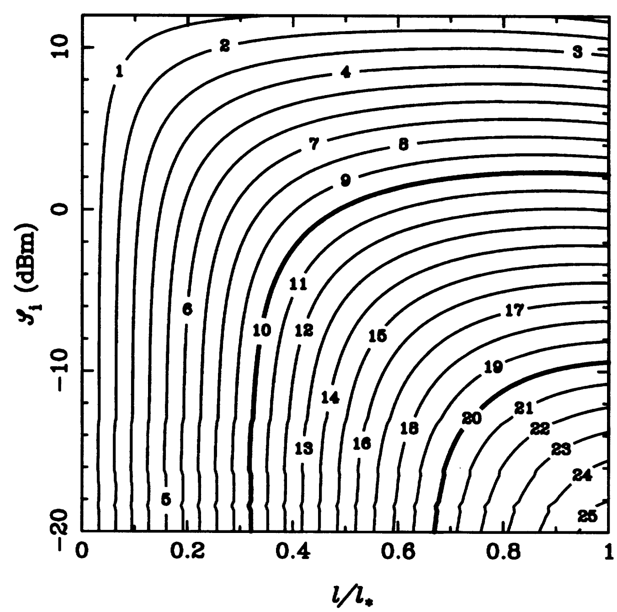

FIG. 3. Level curves (in dB) for the gain as a function of both the amplifier length and input signal (the other parameters are the same as in Fig. 1).

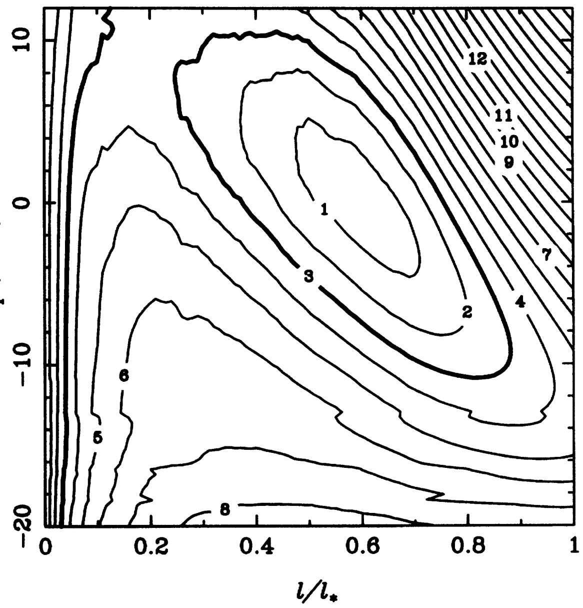

FIG. 4. Level curves (in dB) for the noise figure as a function of both the amplifier and input signal (the other parameters are the same as in Fig. 1).

number approximately given by

$$S_{\text{out}} \simeq 0.7 N_s h\nu \Delta\nu \,. \tag{13}$$

An extensive numerical simulation—for varying parameters N_s, Θ, and σ—shows that Eq. (8) very accurately describes saturation effects in the quantum regime: In fact, for $N_s \gg 1$ and parameters Θ and σ within the validity domain (9), all resulting noise figures are above the Heisenberg limit.

In conclusion, we have shown that a saturable amplifier can beat the 3 dB standard limit of linear amplifiers, approaching the Heisenberg 0 dB limit. The origin of noise suppression is due to saturation itself, which in an intermediate nonlinear regime can effectively cut off the high-intensity tail of the photon distribution, without limiting the gain too much. The best performance is achieved by properly tuning the amplifier length and the input signal, in order that saturation effects are not too strong, with an output signal photon number approximately equal to 0.7 times the saturation photon number. We focused attention on parameters corresponding to an actual amplifier, but a further optimization is possible versus parameters Θ, σ, and N_s, with the aim of achieving larger regions of best performance in the (S_{in}, l) plane: this will be the subject of a forthcoming paper.

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[14] Equation (9) is a consequence of the adiabatic elimination of the atomic variables, which holds in the limit $k < \gamma_\downarrow, \gamma_\uparrow$, where $\gamma_\downarrow$ and $\gamma_\uparrow$ are the transition rates between the two lasing levels [see J. P. Gordon, Phys. Rev. **161**, 367 (1967)]. If inequality (9) is not satisfied, unphysical noise figures lower than unit can be obtained.
[15] For a brief illustration of the Green's-function Monte Carlo simulation method, along with some tests on analytically solvable models, see G. M. D'Ariano, C. Macchiavello, and S. Moroni, Mod. Phys. Lett. B (to be published).