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Formation of Shape-Preserving Pulses in a Nonlinear Adiabatically Integrable System

R. Grobe, F. T. Hioe,* and J. H. Eberly
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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We have obtained a class of solitary wave solutions to novel exactly integrable nonlinear wave equations. Conservation laws can be identified and velocities of propagation predicted. We propose to test our predictions in the optical domain with two-color experiments.

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The propagation of shape-preserving excitations as solutions of nonlinear wave equations is an important subject for a variety of fields of physics, including hydrodynamics [1], particle physics [2], coherent atomic transients [3], plasma physics [4], and optical communications [5]. The balance between nonlinearity and dispersion in nonlinear wave equations is usually so critical for the shape-preserving character of their solutions that only a limited set of discrete solution functions is available, and these are usually restricted to combinations of hyperbolic secants. However, we derive here a pair of nonlinear wave equations that have the unusual property that, as a coupled pair, the need for this critical balance is greatly reduced. We show that they have exact solutions that, for all practical purposes, are *arbitrary* functions. We also give numerical evidence supporting our predictions and identify experimental parameters that appear to permit a two-color optical test of this theory.

Without concern for a specific physical context at the onset, let us begin by writing generic equations for two "fields" interacting with a "medium" in which they propagate. The field amplitudes are denoted V_a and V_b, which depend on laboratory coordinates z and t, and the medium response is determined by two nonlinear functions $f(V_a)$ and $h(V_b)$. In terms of pulse-localized coordinates $\zeta = z$ and $\tau = t - z/c$ the wave operator $\partial/\partial z + \partial/\partial ct$ becomes simply $\partial/\partial\zeta$ and we have

$$\partial V_a/\partial\zeta = igf(V_a), \tag{1a}$$

$$\partial V_b/\partial\zeta = igh(V_b). \tag{1b}$$

Many examples of such nonlinear wave equations are know (sine-Gordon, Korteweg–de Vries, nonlinear Schrödinger, etc.). In almost all if not all such cases in physics, the nonlinear functions f and h are determined by a deeper dynamical interaction that is truncated at the lowest order of nonlinearity, usually cubic. However, we have found a physical example in which this deeper interaction is not truncated, in which exact integrability is nevertheless achieved, and for which the prediction of an arbitrary solution function shape appears to be testable by optical experiments.

First, we explain what we mean by a deeper dynamical interaction. A trivial example occurs when $f(V_a)$ and $h(V_b)$ are obtained from linear equations such as $i\partial f/\partial\tau = -gV_a$. The effect is purely dispersive of course, and causes only a velocity change. However, this example becomes nonlinear in an interesting way if the coupling "constant" g also becomes a dynamical variable that is itself couple to f and h and the V's. Such a situation actually arises naturally in quantum optics where the V's can be interpreted as proportional to the electric field strengths of a pair of laser beams traversing the same medium, and g enters very symmetrically with f and h through the equations

$$i\frac{\partial}{\partial\tau}\begin{bmatrix} f \\ g \\ h \end{bmatrix} = -\begin{bmatrix} 0 & V_a & 0 \\ V_a & i\gamma & V_b \\ 0 & V_b & 0 \end{bmatrix}\begin{bmatrix} f \\ g \\ h \end{bmatrix}. \tag{2}$$

Here $\Psi = [f, g, h]$ is the atomic state vector of three-level atoms comprising the medium, and (2) is the appropriate Schrödinger equation with γ playing the role of an irreversible relaxation rate. We take the initial state vector to be $\Psi(\zeta, \tau = -\infty) = [1, 0, 0]$. Because of this our boundary

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conditions f, h, V_a, V_b are all real and g is pure imaginary throughout. Novel coupled nonlinear waves are predicted by Eqs. (1) and (2), and we have studied them in both transient and shape-preserving domains, finding a number of remarkable solitary wave phenomena. In this Letter we will show that the coupled equations (1) and (2) can be solved without dynamical truncation, i.e., without introducing an approximate nonlinear susceptibility and without relaxation to a steady state.

Let us quickly point out some familiar results if only one wave is present, say V_a, but the dynamical coupling to g is retained. For example, when the single pulse under consideration is slowly changing and weak ($V_a \ll \gamma$), the overdamped solution for g is given by $ig \approx -(V_a/\gamma)f$. When this is substituted into (1a) we get $\partial V_a/\partial\zeta \approx -(1/\gamma)f^2 V_a$. If the pulse is not too long, f can change only negligibly and we get generic exponential-type spatial decay of the pulse as it propagates: $V_a(\zeta,\tau) \approx V_a(0,\tau)\exp[-\zeta/\gamma]$.

Under the opposite assumption (strong pulse: $\gamma \ll V_a$) a soliton solution [6] appears. In this case, by neglecting γ and introducing the integrated-pulse-amplitude variable $\Phi_a(\zeta,\tau) \equiv 2\int_{-\infty}^{\tau} d\tau' V_a(\zeta,\tau')$, we can obtain the sine-Gordon equation of self-induced transparency: $\partial^2\Phi_a/\partial\zeta\partial\tau = \sin\Phi_a$, with its familiar 2π-sech solution. It follows from (1) and (2) that the pulse "energy" $E_a(\zeta) \equiv \int_{-\infty}^{\infty} d\tau' V_a^2(\zeta,\tau')$ satisfies the equation $\partial E_a/\partial\zeta = f^2(\zeta,\tau=\infty) - 1$. The presence of the decay parameter γ in Eqs. (2) mandates that $f(\zeta,\tau=\infty) \to 0$ if the pulse is strong or long enough. Then the pulse energy decreases linearly, not exponentially, with a characteristic decay length Z_a, which can be defined in terms of the pulse energy at $\zeta = 0$, viz., $Z_a \equiv E_a(0)$. Although the existence of Z_a depends on the presence of the relaxation rate γ, the value of Z_a does not [7].

Now we turn to the dynamically coupled two-pulse situation of central interest. Equations (1) and (2) have to be solved consistently as a boundary and initial value problem which, in general, requires a numerical approach. However, we have found analytic solutions under certain conditions. Because an adiabatic assumption is key to the solutions, we use the term "adiabaton" to describe these pulses.

We first present the result of a computer simulation. In Fig. 1(a) we show the temporal profile of the two injected pulses at the entry surface of the medium at $\zeta = 0$: a Gaussian-shaped pulse $V_a(0,\tau) = A\exp[-(\tau/\tau_0)^2]$ and the second pulse with a smooth turn-on and then a constant amplitude $V_b(0,\tau) = B$ (for simplicity). In Figs. 1(b)–1(d) we display temporal snapshots of both pulses after various propagation distances ζ. These solutions were obtained by integrating the five coupled nonlinear partial differential equations (1) and (2) on a space-time grid. We used an iterative predictor-corrector scheme to control the convergence in each space-time propagation step.

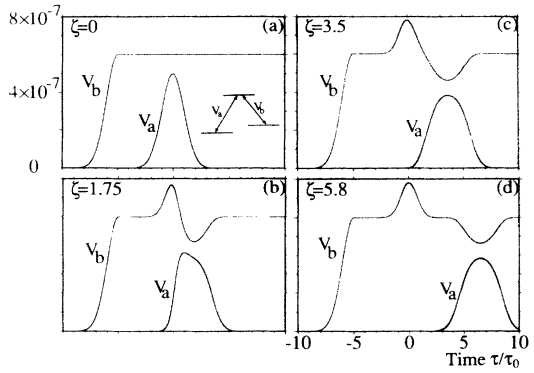

FIG. 1. The evolution of a pulse pair as exact solutions of the nonlinear wave equations (1) and (2) in a medium of three-level atoms in Λ configuration. Temporal profiles of the pulse pairs were taken after various propagation distances ζ (measured in units of Z_a). After $\zeta \approx 3.5$, the pulse reshaping is completed and adiabatons have formed. The bump of V_b at $\tau = 0$ in (b)–(d) travels with the vacuum speed of light c, whereas the adiabaton pair has a reduced speed $u = c/260$. Superimposed on each curve is the analytical solution of Eq. (5). The parameter choice was made with the optical two-color experiment in mind [$\tau_0 = 5 \times 10^7$ a.u., $\gamma = 10^{-7}$ a.u., $A = 5 \times 10^{-7}$ a.u., $B = 6 \times 10^{-7}$ a.u., and $\mu = 6.8 \times 10^{-13}$ a.u. (corresponding to an atomic density $= 1.482 \times 10^{-10}$ a.u. and equal oscillator strength $= 0.1$ a.u. for both transitions)].

The two pulses initially undergo significant reshaping. The field V_a broadens and its amplitude decreases. At the same time V_b develops a dip and a bump [Fig. 1(b)]. When the pulses have propagated further ($\zeta \geq 3.5Z_a$) a stable adiabaton pair has been formed [Figs. 1(c) and 1(d)]. The adiabaton pair (consisting of the dip in V_b and the broadened pulse V_a) travels loss-free distances, which exceed Z_a by orders of magnitude without further change of its shape at a reduced speed.

There are now two theoretical tasks—to predict the nature of this nonlinear shape-preserving two-pulse excitation and to describe the physics of the transients before the fixed pulse shapes are established. Actually, Figs. 1(b)–1(d) already establish that a proper theory of adiabatons can be given, because the same figures also show the analytic solutions we have found, superimposed on the numerical solutions. *The theory makes predictions so close to the numerical solutions that the graphs are practically indistinguishable.* Now we will present the theory and its predictions.

We have found the theory to be tractable under two interesting conditions: First, the fields can be nearly arbitrary as long as they permit the following two-pulse adiabaticity relation to be satisfied:

$$V_b \frac{\partial V_a}{\partial\tau} - \frac{\partial V_b}{\partial\tau} V_a \ll V^3, \tag{3}$$

where V^2 denotes the total pulse energy density, $V^2 \equiv V_a^2 + V_b^2$. Note that it is easy to satisfy (3) by making

either or both of the pulse amplitudes large. Second, the input pulses should be injected in counterintuitive order [8,9], and this mandates that pulse V_b acting on the initially "empty" variables g and h is turned on before pulse V_a, e.g., as shown in Fig. 1(a). Under these two conditions the response of the medium can be very well approximated by the solutions

$$f \approx V_b/V\,, \quad g \approx -\frac{i}{V_a}\frac{\partial}{\partial\tau}\frac{V_b}{V} \approx \frac{i}{V_b}\frac{\partial}{\partial\tau}\frac{V_a}{V}\,, \quad h \approx -V_a/V\,. \tag{4}$$

Given (4), the vector $\Psi(\zeta,\tau) = [f,g,h]$ is one of the three adiabatic eigenvectors of (2) [10]. Because of the counterintuitive pulse turn-on it is the only eigenvector smoothly connected to the initial vector $\Psi(\zeta,\tau=-\infty) = [1,0,0]$. Note that the atomic solutions (4) allow for a significant dynamical rearrangement of atomic level populations.

By inserting solution (4) into the field equations (1), we obtain a novel pair of coupled nonlinear wave equations:

$$\frac{\partial}{\partial\zeta}V_a = -\frac{1}{V}\frac{\partial}{\partial\tau}\frac{V_a}{V}\,, \tag{5a}$$

$$\frac{\partial}{\partial\zeta}V_b = -\frac{1}{V}\frac{\partial}{\partial\tau}\frac{V_b}{V}\,. \tag{5b}$$

Note that the relaxation parameter γ plays no role in these equations. These two equations are nonlinearly coupled through the variable $V(\zeta,\tau) = \sqrt{V_a^2 + V_b^2}$. One can show from (5) that V does not depend on ζ, which means that V is determined by the input fields. If we change from τ to a new variable $Z(\tau) \equiv \int_{-\infty}^{\tau} d\tau' V^2(0,\tau')$, the equations simplify and can be solved analytically. The exact solutions are

$$V_a(\zeta,\tau) = V(0,\tau)\,F_a[Z(\tau) - \zeta] \tag{6}$$

and similarly for V_b. The functional form of F_a is given directly from the input fields via $F_a[x] = V_a[0,Z^{-1}(x)]/V[0,Z^{-1}(x)]$. Here $Z^{-1}(x)$ denotes the inverse function of Z. We point out that because of the nontrivial dependence of Z on τ due to the initial pulse shapes, neither of the fields necessarily propagates with invariant shape even though their individual energies E_a and E_b are conserved.

There are important special cases, such as the one adopted for our numerical example, which allow for a propagation into shape-invariant (adiabaton) pulses. These cases occur if the input fields $V_a(0,\tau)$ and $V_b(0,\tau)$ are chosen such that V is constant after a certain time T. Then, for $\tau \geq T$ the integral $Z(\tau)$ can be solved: $Z(\tau) = B^2\tau + \alpha$, where B is the long-time amplitude of V_b (as in our numerical example) and the constant α is determined by the early pulse forms before T. The time T corresponds to a characteristic propagation distance of $Z(T) = \int_{-\infty}^{T} d\tau' V^2(0,\tau') = Z_a + \int_{-\infty}^{T} d\tau' V_b^2(0,\tau')$ after which the pulse reshaping is completed. This length is required for the initial pulse $V_a(0,\tau)$ to evolve into its shape-preserving adiabaton:

$$V_a(\zeta,\tau) = BF_a[B^2\tau + \alpha - \zeta] \qquad \text{for } \tau \geq T\,. \tag{7}$$

The adiabaton propagates with a laboratory-frame velocity u given by $1/u = 1/c + \mu/B^2$, which may be much smaller than the vacuum speed of light.

In contrast to usual types of steady states which do not depend on initial conditions, adiabatons are specified by the input pulse forms. Different input pulses evolve into different adiabatons as described by Eq. (7). This novel feature is illustrated in Fig. 2. The input field V_a is a sequence of three distinct signals. Each of the signals evolves into its corresponding loss-free propagating adiabaton. Please note that the formed adiabatons do not overlap and that they preserve the distinct "character" of the input signals. The second field V_b (dashed graph) mimics the input pulses which provides for applications in optical transmission a quite useful redundancy.

Predictions arising from our coupled nonlinear wave equations (5), including particularly the arbitrary pulse-shape option, appear to be testable in the optical domain with two-color laser experiments. Many two-field three-level laser propagation experiments [10] have been performed recently in the pump-probe regime ($V_a \ll V_b$), and some misleading similarities exist between adiabatons and matched pulses of electromagnetically induced transparency (EIT). In contrast to the identical envelopes of EIT matched pulses, as discussed in [12], adiabatons are predicted to have the opposite character—complementary envelopes. More important, adiabatons occur in a different parameter regime. The assumption of fully adiabatic behavior, rather than overdamped behavior of g, was the novel step we used to obtain the nonlinear equations (5).

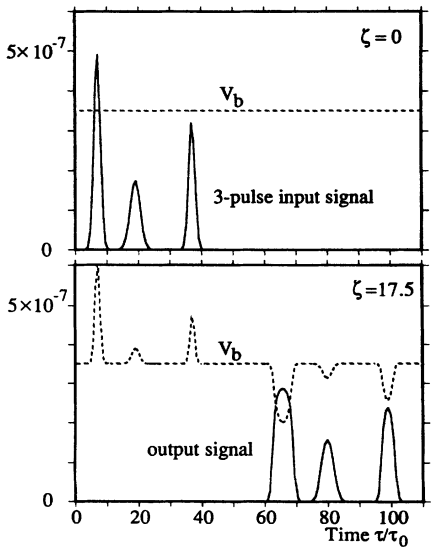

FIG. 2. Propagation of a three-pulse field V_a. Note that the three formed adiabatons (bottom) have a clear correspondence to the input signals (parameters as in Fig. 1).

All length and time scales for EIT pulses depend explicitly on γ, whereas the physics of adiabatons is *independent* of γ or detuning. However, although the phenomena are distinct, the laser frequencies and atomic media used for EIT experiments should also be compatible with adiabaton experiments. It is simply necessary that two-pulse experiments be designed for the case in which the two Rabi frequencies are of comparable strength ($V_a \approx V_b$) such that reshaping effects become important, an accessible regime but not studied to date.

Finally, regarding our proposed optical tests, it is important to recognize that damping processes more general than contained in (2) will actually be present. Therefore, we have checked how our predictions are altered by the inclusion of additional loss mechanisms, such as radiative decay and collisional dephasing of f and h, as well as g. This is possible only by going to a density matrix formalism, and we have done this. In the case radiative decay is included we found that even in cases where the lifetime is as short as the pulse duration, the propagating pulses were almost unaffected. A larger influence on the results could occur due to collisional influences on f and h, if the dipole relaxation times were comparable to the temporal pulse width, but such relaxation times are typically much longer.

To summarize, we have discovered novel two-pulse coupled nonlinear wave equations that are exactly integrable. The solutions, so called "adiabatons," are generated from initially nonideal pulses after a characteristic propagation distance. Adiabatons have the highly unusual property for shape-invariant pulses that, depending only on the input fields at the entry surface of the medium, they can take *arbitrary shapes*. It appears that these novel nonlinear excitations may be detectable in optical two-color experiments.

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*Also with Department of Physics, St. John Fisher College, Rochester, NY 14618.

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