Low-Energy States of Circulating Stored Ion Beams: Crystalline Beams

Jie Wei
Brookhaven National Laboratory, Upton, New York 11973

Xiao-Ping Li*
Rutgers University, Piscataway, New Jersey 08855

Andrew M. Sessler
Lawrence Berkeley Laboratory, Berkeley, California 94720
(Received 21 March 1994)

The low-energy states of a beam of charged particles subject to circumferentially varying guiding and focusing forces is studied by first deriving a Hamiltonian and then using molecular dynamics methods. In an alternating gradient structure, operating below the transition energy (but not in a constant gradient ring), the lowest state is ordered. The nature of the ground state depends upon the beam density and the ring parameters. For very low temperature, the crystal remains intact for a long time, but as the temperature increases it rapidly gains energy from the lattice.

PACS numbers: 41.75.–i, 29.20.Dh, 52.60.+h, 52.65.+z

The ground state of crystalline beams was first studied in seminal work by Schiffer and Rahman [1]. Their work assumed a storage ring model in which charged particles are subject to time-independent harmonic forces in both transverse directions. Subsequently, they studied crystallization in a time-dependent focusing potential which replicated some of the features of alternating gradient focusing and with time-dependent shearing which replicated some of the features of the alternate bending and straight sections of a storage ring [2].

Nevertheless, questions remain about whether or not an ordered state can be created in a real storage ring. Furthermore, with laser cooling very low (longitudinal) temperatures of stored beams have recently been achieved [3]. Thus it is now prudent to develop the tools which will allow one to make calculations that incorporate the characteristics of actual storage rings. We have developed such a formalism, and in this Letter we describe the formalism and then employ it to study the nature of the ordered state in actual storage rings. We find that in operation below the transition energy $\gamma < \nu_x$ alternating gradient (AG) rings, as contrasted with constant gradient rings, can have a crystalline lowest energy—or ground state. This state will change periodically in time, "breathing" as the particles go around the storage ring and are subject to periodic bending, straight sections, focusing lenses, and defocusing lenses. Under some conditions the changes are dramatic, the crystal periodically changing its shape and orientation, but the crystal remains for a very long time in the ground state; i.e., by this process very little heat is put into the crystal (possibly zero at zero temperature). In order to achieve the ordered state, the beam must be very cold; we give results, in typical machine parameters, for just how cold (expressed in terms of energy spread and emittance) the beam must be. We show that there exists a temperature above which the crystal rapidly melts.

In order to use the molecular dynamics (MD) methods we must be in the frame of reference of the particles. That is, a rotating frame (x, y, z, τ) of a reference particle in which the orientation of the axes is rotating so that the axes are constantly aligned to the radial (x), vertical (y), and tangential (z) direction of motion. This is, of course, an accelerating frame of reference. We can derive the equations in the laboratory frame and then transform to the moving frame, but it is most convenient to derive the equations directly in the beam frame, employing the formalism of general relativity [4]. One may think of the result of this process as finding the relativistic generalization of centrifugal and Coriolis forces. In the frame of the reference particle, the particle motion is nonrelativistic. The rather lengthy derivation resulting in a novel Hamiltonian, is given in Ref. [5]; here we only present the results.

It is convenient to scale dimensions in terms of ξ, with $\xi^3 = r_0\rho^2/\beta^2\gamma^2$, where r_0 is the classical particle radius (Z^2e^2/mc^2), Ze is the electric charge, the velocity of a reference particle is βc, its energy is γmc^2, and it moves on an orbit with bending radius ρ in magnetic field B_0. We measure time in units of $\rho/\beta\gamma c$ and energy in units of $\beta^2\gamma^2Z^2e^2/\xi$. In a bend region, with magnetic field B_0, we have the Hamiltonian

$$H = \tfrac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \gamma x P_{z'} + \tfrac{1}{2}x^2 + V_c(x, y, z), \quad (1)$$

where the Coulomb potential

$$V_C = \sum_j [(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2], \quad (2)$$

and the summation j is over all the other particles.

In a straight section, where there is no bending of particles, there often are focusing magnets. If focusing is supplied by a quadrupole of field gradient B_1 so that $B_x = B_1 y$ and $B_y = B_1 x$, then in a straight region, where

0031-9007/94/73(23)/3089(4)$06.00

the focusing strength is characterized by $n = -B_1\rho/B_0$, we have the Hamiltonian

$$H = \tfrac{1}{2}(P_x^2 + P_y^2 + P_z^2) + \tfrac{1}{2}(-nx^2 + ny^2) + V_c(x,y,z). \quad (3)$$

As the particle goes around the storage ring, the appropriate equations of motion must be employed. The change from those corresponding to a curved section to those corresponding to a straight section incorporates, in a quantitatively correct manner, the effect of shear (given by the term $\gamma x P_z$). Similarly the AG effect of alternate gradient is incorporated, quantitatively correctly, by changing the field gradient n (positive for vertical focusing, negative for vertical defocusing, and zero for an open straight section).

Consider a cyclotron magnet, i.e., a magnet that gives constant bending and constant focusing. The Hamiltonian in this case is

$$H = \tfrac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \gamma x P_z + \tfrac{1}{2}(1-n)x^2 + \tfrac{1}{2}ny^2 + V_c(x,y,z). \quad (4)$$

If the gradient of the magnet is such that n lies between 0 and 1 then, as is well-known, the magnet in combination with the centrifugal force gives focusing in both the vertical and horizontal planes. Just the kind of storage rings, one would think, for the formation of a crystal and, yet, that is not true at all. The Hamiltonian in Eq. (4) is bounded from below only if $0 < n < 1 - \gamma^2$. Since $\gamma > 1$, we see that this condition can never be satisfied and, as a result, crystalline beams can never exist in this case. In fact, the centrifugal force becomes defocusing when the particles are crystallized.

Let us now, for pedagogical purposes, separate the effect of "shear" from that of time-dependent focusing. We shall study, as we did in our earlier work [5], an AG ring with a constant bending field. The Hamiltonian is the same as in Eq. (4) but with n time dependent. Assuming there is horizontal and vertical focusing, i.e., $\nu_x > 0$ and $\nu_y > 0$, where ν_x and ν_y are the horizontal and vertical tunes, it is easy to see that the Hamiltonian, in smooth approximation, is bounded from below if and only if $\gamma^2 < \nu_x^2$, and, therefore, crystalline beams exist only if the ring is operated below the transition energy. Our analysis is made for a particular ring, but the criterion that $\gamma^2 < \nu_x^2$ is general.

For a given ring, the nature of the ground state (which term is used to describe the periodic lowest energy state) depends upon the density of particles. When the density is low the ground state is a 1D chain. If the density is larger than the ground state is, a 2D state lies in the plane of weaker focusing, which is determined by whether $\nu_x^2 - \gamma^2$ is greater or smaller than ν_y^2. Notice that the focusing in a crystal is not determined simply by ν_x and ν_y but by the focusing factors $(\nu_x^2 - \gamma^2)^{1/2}$ and ν_y.

The density at which a 1D structure changes into a 2D structure can be determined analytically. It is given by

$$\mathrm{Min}(\nu_y^2, \nu_x^2 - \gamma^2) > 4.2/\Delta_z^3, \quad (5)$$

where Δ_z is the nearest neighbor distance in z (given trivially in terms of the number of particles stored, the circumference of the storage ring, and γ). This relation is similar to that in Ref. [2], but involves the focusing factors. Notice that, in practice, one can change the focusing of a storage ring (changing the operating point) and also change the storage energy. Thus the focusing can be readily changed and the effect of such change easily studied.

In an AG ring, the 2D crystal structure, as contrasted with the 1D structure, will "breath" as the particles go about the storage ring. Despite this motion little energy is pumped into the crystal; it remains in its ground state for a very long time. Such behavior is not unexpected, for as particles go around the AG storage rings, the amplitude of their oscillation changes (β-function variation), and particles of different energy move closer or further apart (η-function variation), yet particles can be stored forever [6].

In general, when Coulomb interactions and AG focusing are present it is impossible to solve the equations analytically; we obtain numerical solutions using MD. This method allows us to determine the lowest energy state in realistic storage rings (that is, the actual ring lattice can be inserted in the computation) and also allows us to study behavior as the crystal temperature is increased from absolute zero. We can also determine the temperature at which such a crystal melts, i.e., loses its long-range order as the particles go into a state in which they pass each other (as in a usual storage ring).

In the MD calculations we choose time and space steps, the number of particles in a cell, and the length of cells so that the results are independent of these choices. In finding the ground state we "cool" once per lattice period by simply imposing a condition of periodicity (by averaging initial and final coordinates and momenta), while correcting P_z according to the amount of slippage in z for each particle and for many periods (typically 1000) and then turning the cooling off and observing no change (in one case for up to 10^6 periods). The results are independent of initial conditions. We have checked that "cooling," which better replicates the actual experiments, leads to the same state (but takes very long when the density is high).

We find, numerically, that when the density is higher than that appropriate to a 2D crystal, the particles arrange themselves into 3D crystals. For even higher density, the crystals become helices and then helices within helices. An example is shown in Fig. 1. These structures are similar to, but differ in detail, from that given in Ref. [1]. It is seen that the interparticle spacing in all these structures is approximately the same and given (roughly) by the interparticle spacing when transition is made from 1D crystal to a 2D crystal, i.e., can be characterized by ξ, ν_x^2 and ν_y^2, except for dimensionless numbers near unity. Thus, the crystal forms cylindrical shells within cylindrical shells, upon which the particles are deployed in such a way that the interparticle spacing is about the same. This behavior is very similar to that which occurs in ordinary crystals. It appears to be the case that a crystal (with many shells) forms no matter how high the particle density is.

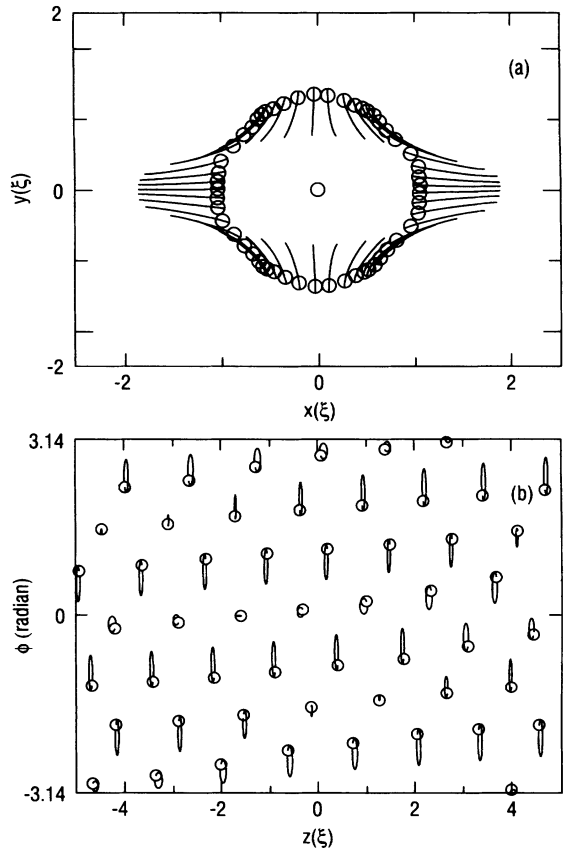

FIG. 1. A 3D structure with particle positions projected (a) into the x-y plane and (b) into the ϕ-z plane, where ϕ is the polar angle. The lattice is a FODO lattice with constant bending with $\nu_x = 2.7$ and $\nu_y = 2.3$, and the particle energy is $\gamma = 1.4$. The total number of particles is 60, and the MD period length is 10ξ. The particles move periodically in time, with the solid lines showing their trajectories and the circles indicating their position at the start and end of the each lattice period.

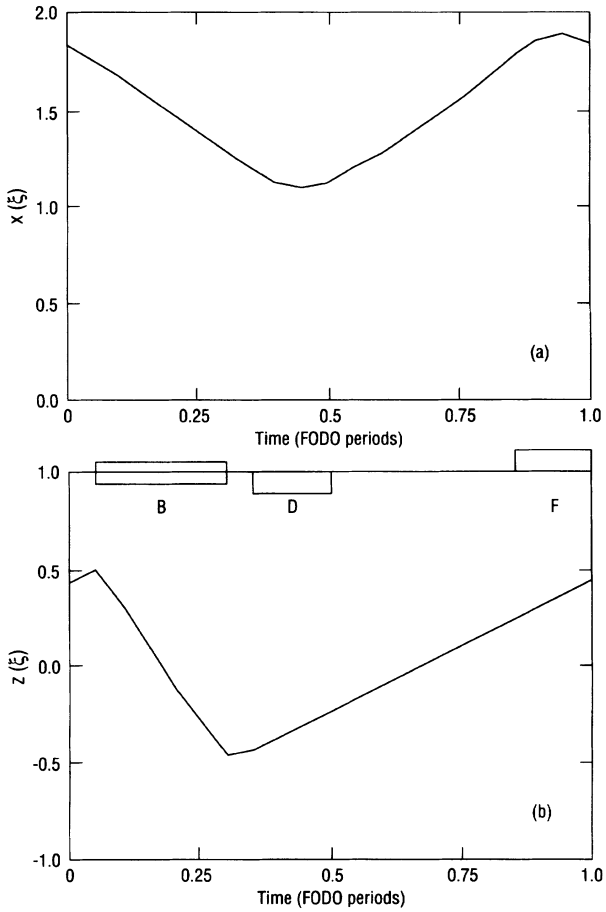

FIG. 2. The effect of shear. In this study $N = 40$, $L = 40\xi$. The cell of one of the particles with largest horizontal displacement (and no vertical displacement) is shown. Motion occurs both (a) in the x direction (breathing) and (b) in the z direction (shear). Lattice components in one of the 10 periods are displayed on the figure: B is a bend section; F is a focusing section; and D is a deforming section.

Having determined that ground states can be formed in a storage ring with AG (time-dependent) focusing, we now study the effect of shear upon the ground state, that is, the effect of time varying bending. To do this in a specific case, we took an AG lattice having 10 periods of the FODO type (focusing, open or drift, defocusing, open or drift) where with constant bending a ground state crystal can exist. The bending was then concentrated into a small region of the period. The result of this is shown in Fig. 2. It can be seen that the crystal "takes up the difference" between constant angular velocity and constant linear velocity by adjusting the spacing between particles, i.e., by converting potential energy into kinetic energy. This result is generally true in all cases we have studied, i.e., shear does not seem to destroy a crystal, but may affect its melting temperature.

Having studied the ground state, we now investigate behavior at nonzero temperatures [7]. Temperature is defined in terms of the deviation of P_x, P_y, and P_z from that of the ground state, then squared and averaged over a period and over all particles. The temperature can readily be expressed in terms of the usual accelerator parameters of un-normalized emittance, ε and relative momentum spread $\Delta p/p$ by $\Delta\varepsilon_x = \xi^2\beta_x T_x/\rho^2$, $\Delta\varepsilon_y = \xi^2\beta_y T_y/\rho^2$, and $\Delta p/p = \xi\gamma T_z^{1/2}/\rho$. The Hamiltonian for particles in a storage ring is quite different from the conventional Hamiltonians encountered in condensed matter physics. It is time dependent, and, therefore, energy is not conserved. Furthermore, the Hamiltonian without the smooth approximation is not bounded from below at any time, and the adiabatic approximation cannot be used. It is precisely the dynamical coupling between the external focusing-defocusing force and the Coulomb interaction among the particles that gives us the well-defined periodic structures which we call the "ground state." However, the time dependence of the Hamiltonian can cause damage to the crystal, because energy can be pumped into the system by creating phonons. As a result, a crystalline beam, especially at nonzero temperatures, cannot last forever unless energy is pumped out of the system by means of a refrigerator.

The frequency of the machine lattice must be several times higher than the highest phonon frequency in the system, so that the transfer of energy into the system can only be realized by multiphonon emission. This process is expected to be strongly temperature dependent. We have studied the relation between the heating rate and the temperature. Figure 3 shows the survival time of a 3D crystalline beam, subject to shear and without cooling, as a function of the initial temperature at which a MD run starts. The survival time indeed strongly depends on the initial temperature and rapidly tends to a large value when the initial temperature is low enough. This indicates that at low temperature the rate that energy transfers into the system is very low, and it is easy to maintain a crystalline beam for a long time.

Finally, it is well known from the work of Landau and Lifshitz [8] and the work of Emery and Axe [9] that no sharp phase transition exists in one-dimensional systems. The systems we study are all one dimensional in the sense that in the two directions perpendicular to the beam, the systems are always finite, and therefore, we might not expect sharp phase transitions between these phases. However, two qualitatively different states exist in these systems, one is a low temperature "condensed state" (crystalline beam) in which there is limited shearing motion in the z direction, but no passing of particles, and the other is a high temperature "running state" in which particles shear relative to each other. Our numerical studies, to the accuracy of the computer, suggest that the transition between these two states is sharp.

We want to acknowledge the assistance of R. W. Hasse in making our computer program more efficient. We had useful conversations with J. S. Hangst, D. Habs, J. P. Schiffer, V. J. Emery, and V. V. Avilov. This work was performed under the auspices of the U. S. Department of Energy, Office of Energy Research, Office of High Energy and Nuclear Physics under Contract No. DE-AC03-76SF00098, as well as supported by NSF Grant No. DMR-91-15342.

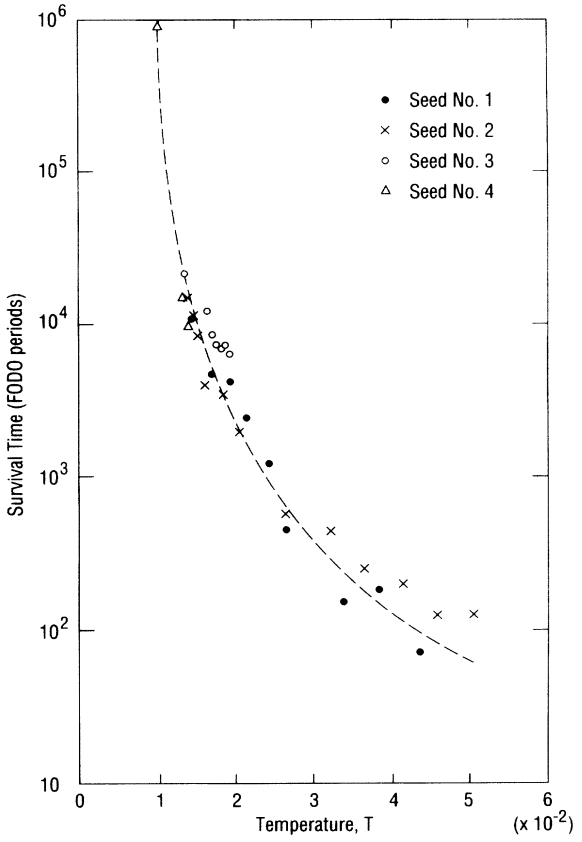

FIG. 3. A study of survival time (time before particles slip past each other) of a crystal in the absence of cooling, as a function of the initial temperature of the crystal. For these studies the same lattice was employed as in Fig. 2. The parameters were $\rho = 1m$, $\gamma = 1.4$, $\nu_x = 2.8$, and $\nu_y = 2.1$. In all cases $N = 40$ and $L = 40\xi$. More than one point, at the same temperature, is due to different random seeds used to distribute initial particle position and velocity appropriate to the chosen temperature.

*Present address: BIOSYM Technologies Inc., 9685 Scranton Rd., San Diego, CA 92121.

[1] A. Rahman and J. P. Schiffer, Phys. Rev. Lett. **57**, 1133 (1986).

[2] J. P. Schiffer and A. Rahman, Z. Phys. A. **331**, 71 (1988); J. P. Schiffer, in Proceedings of the Workshop on Crystalline Ion Beams (Werheim, Germany, 1988), edited by R. W. Hasse, I. Hofmann, and D. Liesenp. 2; J. Beebe-Wang, N. Elander, and R. Schuch, Nucl. Instrum. Methods **B79**, 806 (1993).

[3] S. Schroder et al., Phys. Rev. Lett. **64**, 2901 (1990); J. S. Hangst et al., Phys. Rev. Lett. **67**, 1238 (1991).

[4] C. Møller, *The Theory of Relativity*, (Clarendon Press, Oxford, 1952).

[5] J. Wei, X-P Li, and A. M. Sessler, in *Proceedings of the 1993 Particle Accelerator Conference, Washington, 1993* (IEEE, Piscataway, 1993), p. 3527, and more extensively as Brookhaven National Laboratory Report No. BNL-52381, 1993.

[6] With $N = 100$ and $L = 50$ and a FODO lattice, the ground state is a string on an axis plus one shell containing most of the particles. We ran the MD calculation for 40 000 steps, or 2000 structure periods, and saw no change. We then made a very long run for a lower density case (but still one shell) $N = 40$ and $L = 40$. After 20×10^6 steps, or 1×10^6 structure periods, we could discern no change. We also studied a higher density case, where the ground state consists of five shells, for 20 000 steps, for 1000 structure periods. Again we saw no change in time of the ground state.

[7] J. Wei, X.-P. Li, and A. M. Sessler, in *Proceedings of the Workshop on Beam Cooling and Related Topics, Montreux, 1993* (CERN, Geneva, 1994), p. 482.

[8] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958), p. 482.

[9] V. J. Emery and J. D. Axe, Phys. Rev. Lett. **40**, 1507 (1978).