Relativistically Covariant Symmetry in QED

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We construct a relativistically covariant symmetry of QED. Previous local and nonlocal symmetries are special cases. This generalized symmetry need not be nilpotent, but nilpotency can be arranged with an auxiliary field and a certain condition. The Noether charge generating the symmetry transformation is obtained, and it imposes a constraint on the physical states.

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Quantum gauge theory is founded on phase symmetry, but gauge degrees of freedom bring in extra independent variables. One introduces gauge conditions to suppress these variables but destroys the gauge symmetry thereby. In path integral form, with the introduction of ghosts, gauge invariance is recovered through the passage to the Becchi-Rouet-Stora-Tyutin (BRST) cohomology [1]. The BRST theory raises the ghosts to a prominent role for it regards all fields, including ghosts, as elements of a single geometrical object, the cohomology.

Since locality has been argued to be the main cause of infinities in the usual quantum field theory, people have been turning to nonlocal quantum field theory [2,3]. Nonlocal gauge symmetry plays an important role in nonlocal quantum field theories.

The recent work of [4] Lavelle and McMullan (LM) ingeniously reveals that local QED exhibits a nonlocal symmetry, here called the LM symmetry, which is nilpotent but not Lorentz covariant. They dealt with the following Lagrangian with a gauge fixing term and ghosts $C(x), \overline{C}(x)$ [5]:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overline{\psi}(i\gamma_\mu D^\mu - m)\psi + i\overline{C}\,\partial_\nu\partial^\nu C\,, \qquad (1)$$

where $D_\mu = \partial_\mu - ig_0 A_\mu$. The nonlocal LM transformation is

$$\delta A_0 = i\overline{C}, \qquad \delta A_i = i\tfrac{\partial_i\partial_0}{\nabla^2}\overline{C},$$
$$\delta C = -A_0 + \tfrac{\partial_i\partial_0}{\nabla^2}A_i + \frac{g_0}{\nabla^2}\overline{\psi}\gamma_0\psi, \quad \delta\overline{C} = 0, \qquad (2)$$
$$\delta\psi = [\tfrac{g_0}{\nabla^2}\partial_0\overline{C}]\psi, \qquad \delta\overline{\psi} = \overline{\psi}\tfrac{g_0}{\nabla^2}\partial_0\overline{C}.$$

Covariance is not manifest in the above equations. The operator $1/\nabla^2$ makes the LM transformation nonlocal. The LM symmetry leads to the existence of a nonlocal fermionic Noether current and a corresponding Noether charge, which generates the LM transformation. It also imposes a constraint condition on the physical states besides the usual BRST symmetry [5].

Usually we seek Poincaré-covariant symmetries in gauge theory. In fact, Eq. (2) can be reexpressed in the following (still not Poincaré-covariant) form, with the aid of the equations of motion for the A_μ and C, namely, on shell:

$$\delta A_\mu = i\partial_\mu(\tfrac{\partial_0}{\nabla^2}\overline{C}),$$
$$\delta C = -\tfrac{\partial_0}{\nabla^2}\partial_\mu A^\mu, \qquad \delta\overline{C} = 0, \qquad (3)$$
$$\delta\psi = g_0(\tfrac{\partial_0}{\nabla^2}\overline{C})\psi, \qquad \delta\overline{\psi} = \overline{\psi}\,\tfrac{g_0}{\nabla^2}\partial_0\overline{C}.$$

In fact, if choosing the Feynman gauge, i.e., $\xi = 1$ in Eq. (3), one can verify that action is invariant under Eq. (3) without using the equations of motion, that is, Eq. (3) represents another kind of nonlocal symmetry existing in QED, which is equivalent to LM only on shell. This symmetry is nilpotent, and it too should impose a constraint on the physical states besides the BRST and LM symmetries. With the interchange $C \to i\overline{C}$ and $\overline{C} \to iC$, one can obtain the antiform of the symmetry defined by Eq. (3).

LM's work and Eq. (3) show that we do not have the full story of symmetry in gauge theory, even in QED. In this Letter we demonstrate that there exists a more general Poincaré-covariant symmetry in QED, which includes the local and nonlocal symmetries already mentioned. The symmetry is not nilpotent in general, but it becomes nilpotent under certain conditions.

In the following we consider only operators $\hat{\Omega}$ that are sufficiently "regular" in the sense that they possess adjoint $\hat{\Omega}^\dagger$ with

$$\int_{-\infty}^{+\infty} d^3x\ \phi(\hat{\Omega}\varphi) = \int_{-\infty}^{+\infty} d^3x\ (\hat{\Omega}^\dagger\phi)\varphi \qquad (4)$$

under proper boundary conditions of ϕ and φ, in which the sign † represents Hermitian conjugation. Examples: ∂_μ, ∇^2, $1/\nabla^2$ [4].

The Lorentz and Coulomb gauges are often used; their equivalence is easily proved in path integral form. Being Poincaré covariant, the Lorentz gauge is preferred in path integral formulations, in view of Eq. (1). Accordingly, we concentrate our studies on Poincaré-covariant symmetries of QED in this paper. We consider a Poincaré-covariant generalization of Eq. (2) of the form

$$\delta A_\mu = \partial_\mu(fC + g\overline{C}), \qquad (5)$$

where f and g are fermionic operators, that is, include Grassmann constants. In addition, f and g commute with ∂_μ:

$$\partial_\mu(f, g) = (f, g)\partial_\mu\,. \qquad (6)$$

When A_μ transforms by Eq. (5), the transformations of $C, \overline{C}$ and ψ that leave the action S invariant are

$$\begin{aligned} \delta C &= \tfrac{i}{\xi} g^\dagger \partial_\mu A^\mu, \\ \delta \overline{C} &= -\tfrac{i}{\xi} f^\dagger \partial_\mu A^\mu, \\ \delta \psi &= -i g_0 (f C + g \overline{C}) \psi, \\ \delta \overline{\psi} &= i g_0 \overline{\psi} (C f^\dagger + \overline{C} g^\dagger), \end{aligned} \tag{7}$$

in which f and g are regular in the sense of Eq. (4). One may conclude that even if Eqs. (5) nonlocal transformation, f and g will not alter the action S between the end points of the integration over space; see, for example, Ref. [4]. Thus, Eq. (5) and Eq. (7) actually represent a symmetry of QED.

In this generalized transformation, the unique requirement on f and g is that they should be regular operators in the sense of Eq. (4). It is easily checked that the BRST symmetry, the symmetry of Eq. (3), and their antiforms are all special examples of this more general symmetry. In the following we study some properties of this symmetry.

The generalized symmetry need not be nilpotent in general; see, for example, $f = \lambda_1, g = \lambda_2, \lambda_1 \neq \lambda_2$. Nilpotent symmetries such as BRST define a cohomology but our more general symmetry does not. Moreover, the non-nilpotent transformation defined by Eqs. (5) and (7) exhibits the commutation relations of super-Lie algebra.

However, our generalized symmetry is nilpotent under the following conditions. For A_μ, one can verify that the following condition leads to $\delta^2 A_\mu = 0$ from Eqs. (5) and (7):

$$f g^\dagger = g f^\dagger. \tag{8}$$

This condition is evidently fulfilled in BRST symmetry and that of Eq. (3), since one of f, g is zero in those cases.

For $C, \overline{C}$, we see that $\delta^2 = 0$ generally holds only on shell. In order to have a "strong" nilpotency in the theory in the sense that $\delta^2 (C, \overline{C}) = 0$ off shell and on, we add an auxiliary term $\frac{1}{2} E^2$ to the Lagrangian of Eq. (1), where E is a bosonic field. Then, the transformation

$$\begin{aligned} \delta C &= \tfrac{i}{\xi} g^\dagger \partial_\mu A^\mu - \tfrac{i}{\sqrt{\xi}} g^\dagger E, \\ \delta \overline{C} &= -\tfrac{i}{\xi} f^\dagger \partial_\mu A^\mu + \tfrac{i}{\sqrt{\xi}} f^\dagger E, \\ \delta E &= \tfrac{1}{\sqrt{\xi}} \partial_\mu \partial^\mu (f C + g \overline{C}) \end{aligned} \tag{9}$$

fixes the action S (with the auxiliary term added in) and also leads to $\delta^2 (C, \overline{C}, E) = 0$, where A_μ transforms still according to Eq. (5). It is easy to check that $\delta^2 A_\mu = 0$ still holds under transformation (9).

Thus we have obtained a generalized symmetry of QED, represented by Eqs. (5) and (9), which is relativistically covariant and nilpotent and includes both local and nonlocal forms.

The transformations (5) and (9) have an evident additive group structure. Therefore we take it for granted that there is an interpolation between the BRST symmetry and that of Eq. (3). Specifically, if f and g take the values

$$f = \lambda_1, \qquad g = -i \frac{\partial_0}{\nabla^2} \lambda_2, \tag{10}$$

one can easily check that Eqs. (5) and (9) express exactly this interpolation, which is still a nilpotent transformation. We can construct various symmetries of QED by selecting f and g.

The following Noether charge generates the transformation Eqs. (5) and (9):

$$\begin{aligned} Q = i \int d^3x \bigg\{ & \partial_\mu (f C + g \overline{C}) \left[\partial_0 A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu A_0 \right] \\ & - \frac{1}{\xi} [g^\dagger (\partial_\mu A^\mu - \sqrt{\xi} E)] (\partial_0 \overline{C}) \\ & - \frac{1}{\xi} [f^\dagger (\partial_\mu A^\mu - \sqrt{\xi} E)] (\partial_0 C) \bigg\}. \end{aligned} \tag{11}$$

If f, g do not depend on time, then Q becomes

$$\begin{aligned} Q = \int i d^3x \bigg\{ & - \left[\partial_0 \partial_i A^i - \left(1 - \frac{1}{\xi}\right) \nabla A_0 \right] (f C + g \overline{C}) \\ & - \left(\frac{1}{\xi} \partial_i A^i - \frac{1}{\sqrt{\xi}} E \right) \partial_0 (f C + g \overline{C}) \bigg\}. \end{aligned} \tag{12}$$

The nilpotency of the transformations implies $Q^2 = 0$. The charge is anti-Hermitian and is the foundation of the cohomology of the generalized symmetry. Since f, g may be operators generating nonlocal symmetries, it is useful to extend the usual cohomology to a nonlocal form. This work is not contained in this paper.

The physical fields must be invariant under generalized symmetry of Eqs. (5) and (9). Accordingly, the physical states $|\Psi\rangle$ satisfy

$$Q |\Psi\rangle = 0. \tag{13}$$

Evidently, this constraint on the physical states covers many special constraints such as BRST and Eq. (3). In this sense, the condition Eq. (11) is stronger.

In conclusion, we have exhibited a relativistically covariant symmetry of QED that covers and generalizes various local and nonlocal symmetries, including the Eq. (3), BRST, and their antiforms. This generalized symmetry need not be nilpotent, but becomes nilpotent under a certain condition and with the introduction of an auxiliary field. Evidently QED has new non-nilpotent symmetries. The symmetry imposes a constraint on the physical states, which determines the physical states more strongly than previous symmetries such as the BRST. We should note that LM symmetry, Eq. (2), is not covariant except on shell, so it is not included in Eqs. (5) and (7) strictly. A larger class of symmetry including covariant and noncovariant forms is worthy of investigation.

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