Elementary Excitations of One-Dimensional t-J Model with Inverse-Square Exchange

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We identify exact excitation content of the intermediate states for the one-particle Green's functions, spin-spin, and (charge) density-density correlation functions of the periodic one-dimensional t-J model with inverse-square exchange. The excitations consist of neutral $S = 1/2$ spinons and spinless (charge $-e$) holons with semionic fractional statistics and bosonic (charge $+2e$) "antiholons" which are excitations of the holon condensate. We find a set of selection rules and the regions of nonvanishing spectral weight in the energy-momentum space for the various correlation functions.

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Recently, there have been many developments in understanding the family of Calogero-Sutherland models (CSM) which are identified with their peculiar inverse-square exchange (ISE) [1–8]. An important feature of these models is that they belong to the same low-energy universality class as the family of Bethe-ansatz solvable models and may provide a new fully soluble paradigm next to the noninteracting models [2].

The one-dimensional supersymmetric ISE t-J model [3] represents a fixed point model where the elementary excitations form an ideal gas obeying fractional statistics. In contrast to this model, the nearest neighbor exchange (NNE) t-J model [9,10], which has essentially the same low-energy spectra spanned by the same elementary excitations, obscures the simple low-energy structure intrinsic to this class of models. We rediscover the spinons, the holons, and the antiholons—the elementary excitations of the NNE t-J model [10]—in the context of the supersymmetric Yangian of the ISE model. Furthermore, we find that only the states with *finite* number of these elementary excitations contribute to the spectral functions of the one-particle Green's functions ($G^{(1)}$), the charge density-density ($C^{(c)}$) and the spin-spin correlation functions ($C^{(s)}$).

First, we examine the symmetry in the ISE supersymmetric t-J model. The model with periodic boundary conditions possesses, in addition to the global $SU(m|n)$ supersymmetry, a hidden dynamical "quantum group" symmetry algebra called the supersymmetric Yangian [2,4,11]. This symmetry is responsible for the "supermultiplets" in the energy spectrum and the ideal gaslike features of the elementary excitations and, furthermore, provides us with a simple numerical way to identify the exact content of the elementary excitations relevant for the various correlation functions.

The supersymmetric generalization of the $SU(n)$ Haldane-Shastry model Hamiltonian [5–7] is given by

$$H = t \sum_{i<j} \frac{P_{ij}}{d^2(n_i - n_j)}, \qquad (1)$$

where $d(x) = (N_a/\pi)\sin(\pi x/N_a)$ and N_a is the total number of sites. If $a_{i\alpha}^\dagger$ ($a_{i\alpha}$) creates (destroys) a particle of species α at site i and satisfies the single occupancy condition, $\sum_\alpha a_{i\alpha}^\dagger a_{i\alpha} = 1$, the exchange operator can be written as $P_{ij} = \sum_{\alpha\beta} a_{i\alpha}^\dagger a_{j\beta}^\dagger a_{i\beta} a_{j\alpha}$. If m of the species labeled by α are bosons, and n are fermions, the model (1) has a global $SU(m|n)$ supersymmetry with generators given by the traceless part of $J_0^{\alpha\beta} = \sum_i a_{i\alpha}^\dagger a_{i\beta}$. The Yangian symmetry generator of the periodic ISE model is

$$J_1^{\alpha\beta} = \sum_{i>j,\gamma} w_{ij} a_{i\alpha}^\dagger a_{j\gamma}^\dagger a_{i\gamma} a_{j\beta}, \qquad (2)$$

where $w_{ij} = \cot[\pi(i - j)/N_a]$. The higher order generators of the Yangian are obtained recursively from various commutators involving only J_0 and J_1 [4,11].

If we specialize to $SU(1|2)$ supersymmetry, with $\alpha \in \{0, \uparrow, \downarrow\}$, we can rewrite the Hamiltonian in terms of the $SU(2)$ fermionic operators $c_{i\sigma}^\dagger = a_{i\sigma}^\dagger a_{i0}$ as $\mathcal{P}H^0\mathcal{P}$, where H^0 (up to a shift in total energy and in chemical potential) is

$$-\sum_{i\neq j,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{i<j} (J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j + V_{ij} n_i n_j), \qquad (3)$$

where $t_{ij} = J_{ij}/2 = -2V_{ij} = t/d^2(i - j)$ and $n_i = n_{i\uparrow} + n_{i\downarrow}$; $\mathcal{P}$ is the projection operator that projects out all states with doubly occupied sites. The ground state $|\Psi_0\rangle$ of this model is known [3,6] to be

$$\sum_{\{x,\sigma\}} \prod_{i>j} (z_i - z_j)(i)^{\mathrm{sgn}(\sigma_i - \sigma_j)} \prod_k z_k^{J_0} \prod_j c_{x_j\sigma_j}^\dagger |0\rangle, \qquad (4)$$

where $z_j = \exp(i2\pi x_j/N_a)$, $J_0 = -(N/2 - 1)/2$, N is the total number of particles, and $|0\rangle$ the electron vacuum (empty state). In order to have a nondegenerate ground state, we take $N/2$ to be odd. Note that this wave function is just the full Gutzwiller projection of a free electron state [12].

A remarkable feature of this model is that the eigenstates of (1) form degenerate supermultiplets [5] with multiplicities much higher than those expected from the global supersymmetry. All supermultiplets on the $SU(m|n)$ model with $m, n > 0$ are present (with the same

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energy and momentum, but multiplicity reduced to 2) in the spinless free fermion SU(1|1) model [2]. This means that they can be represented by a binary sequence of $N_a - 1$ ones and zeros, representing (in the spinless fermion model) the occupations of Bloch states with nonzero momentum (the zero-momentum orbital has zero energy, which is the supersymmetry, and its occupation is not fixed). There are thus 2^{N_a-1} distinct supermultiplets.

In the SU(1|2) case, the "occupation number" sequence describes a supermultiplet spanning a large range of possible fermion charges N. The state of minimum charge in the supermultiplet is given by the number of zeros in the sequence; the maximum charge is N_a minus the number of times two consecutive ones occur. The ground state of the model with $t > 0$ has a sequence $111\ldots111$, so its minimum charge is $N = 0$ and its maximum charge is $N_a - (N_a - 2) = 2$. The multiplet represented by the alternating sequence $10101\ldots10101$ has a maximum charge state $N = N_a$, which is the spin-singlet ground state of the antiferromagnetic $S = 1/2$ Haldane-Shastry chain, and a minimum charge $(N_a - 2)/2$.

We study the model (1) with $t > 0$ and a chemical potential that *maximizes* N, so the ground state has $0 < N < N_a$. Then, only intermediate states with the *maximum* value of charge in their supermultiplet contribute to the thermodynamic limit of the ground-state correlation functions. To determine the excitation content of these maximal charge states, it is convenient to add a zero to each end of the binary sequence, expanding its length to $N_a + 1$. The ground-state sequence is then of the form $0101010\ldots1111111\ldots0101010$, with a central section of consecutive ones, with equal-length wings of the alternating sequence.

In the limit $N = N_a$, the excitations of the $S = 1/2$ antiferromagnet are neutral spin-1/2 *spinons* [13–15] represented by consecutive zeros (e.g., $\ldots01010010101\ldots$) and spinless charge $-e$ *holons* by consecutive ones (e.g., $\ldots010101101010\ldots$). At intermediate densities, the central region $\ldots1111111\ldots$ may be considered as a *holon condensate* or "pseudo Fermi sea." However, the holons and spinons are *not* fermions, but *semions,* or particles with "half-fractional" statistics, resulting from the spin-charge separation of a hole, which is a spin-1/2, charge $-e$ fermion. A configuration $\ldots11111110111111\ldots$ has a *"hole in the holon condensate"* which we will call an "antiholon"; because of the semionic statistics of the holons, we identify it as a charge $+2e$, spinless boson.

Using concepts from Chern-Simons theory, as applied to the fractional quantum Hall effect [16], if condensed particles have charge q and statistics $\Theta = \pi\lambda$, vortices or holes in the condensate have charge $-q/\lambda$, and statistics $\Theta' = \pi/\lambda$. Here holons have charge $-e$ and $\Theta = \pi/2$ (a semion), so the vortex or hole in the holon condensate (antiholon) then has charge $2e$ and $\Theta = 2\pi$ (a boson). The applicability of such "2D" concepts to 1D ISE-type models has recently been demonstrated: The holon (antiholon) corresponds to particle (hole) excitations of the $\lambda = 1/2$ Calogero-Sutherland model where the particle excitations are semions and the holes $\lambda = 2$ bosons [2,17].

The main results of this Letter can be summarized in Table I, which lists all the possible elementary excitations for the corresponding local perturbations of the ground state. The quantum symmetry prevents the injected electron or hole from breaking up into more than a very simple set of elementary excitations consisting of the left (right) spinons ($s_{L(R)}$), holons ($h_{L(R)}$), and antiholons ($\bar{h}$). As a result, the spectral functions of the various dynamical correlation functions vanish except in certain regions of the energy-momentum plane (i.e., has *"compact support"*).

Figures 1–3 show the regions of compact support formed by the *finite* number of elementary excitations contributing to the intermediate states for $G^{(1)}$, $C^{(c)}$, and $C^{(s)}$, respectively. If the correlation functions are given by the following integral:

$$C(x,t) = \int_{(Q,E)\in\sigma} dQ\, dE\, S(Q,E) e^{i(Qx-Et)}. \tag{5}$$

The figures show the region σ where the spectral function $S(Q,E)$ is nonzero; this is determined by combining the energies and (Bloch) momenta of the finite set of elementary excitations contributing to $S(Q,E)$.

The dispersion relations for the spinon, holon, and antiholon in the thermodynamic limit are $E_{s_{R(L)}}/t = -q(q \mp v_s^0)$, $E_{h_{R(L)}}/t = q(q \pm v_c^0)$, and $E_{\bar{h}}/t = (v_c^0)^2 - q/2$, respectively, where $v_s^0 = \pi$ (spin-wave velocity), $v_c^0 = \pi(1 - \bar{n})$ (sound velocity), and $\bar{n}$ the density of electrons. The right (left) spinons and holons take the upper (lower) signs and are allowed only in $0 \le q \le \pi\bar{n}/2$ ($-\pi\bar{n}/2 \le q \le 0$) relative to the $Q = 0$ ground state, while the antiholons propagate in region $-v_c^0 \le q \le v_c^0$. The curvature of the antiholon dispersion is half that of the holon, indicating that $\bar{h}$ is made by destroying two

TABLE I. List of all the possible excitations from the ground state perturbed by the local operators $c_{i\sigma}(c_{i\sigma}^\dagger)$ ($G^{(1)}$), $n_{i\uparrow} + n_{i\downarrow}$ ($C^{(c)}$), and $n_{i\uparrow} - n_{i\downarrow}$ ($C^{(s)}$). The mirror states ($L \leftrightarrow R$), not listed, are also allowed. The spinon (v_s), holon (v_h), antiholon ($v_{\bar{h}}$), spin-wave (v_s^0), and sound (v_c^0) velocities always satisfy (i) $v_c^0 < v_s^0$, (ii) $v_c^0 \le |v_h|(|v_s|) \le v_s^0$, (iii) $|v_{\bar{h}}| \le v_c^0$, and (iv) for a given spinon-holon pair (s_R, h_R), $|v_{s_R}| \ge |v_{h_R}|$.

Local operator $\hat{O}_i$	Excitation contents of $\hat{O}_i\lvert\Psi_0\rangle$
$c_{i\sigma}$	$(s_L, h_L) + \bar{h} + 2(s_R, h_R)$
$c_{i\sigma}^\dagger$	$(s_L, h_L) + \bar{h}$
$n_{i\uparrow} + n_{i\downarrow}$	$(s_L, h_L) + \bar{h} + (s_R, h_R)$
	$\bar{h} + 2h_R$
$n_{i\uparrow} - n_{i\downarrow}$	$(s_L, h_L) + \bar{h} + (s_R, h_R)$
	$2s_L$

holons. It is then natural to assign charge $C = +2e$ and $S = 0$ to the antiholon while $C = 0$ and $S = \frac{1}{2}$ to the spinon, and $C = -e$ and $S = 0$ to the holon. This assignment is consistent with the results given in Table I and the phase shift calculations. In fact, using this charge conservation argument we were able to identify one extra right holon for the local hole excitation ($\hat{O}_i = c_{i\sigma}$) in Table I, which could not be resolved numerically because of the small system size ($N_a = 12$) studied.

We outline below how to find the regions of support for the various correlation functions. First, we numerically find all the eigenstates having nonzero overlap with the states $c_{i\sigma}(c_{i\sigma}^\dagger)|\Psi_0\rangle$ (for $G^{(1)}$), $(n_{i\uparrow} + n_{i\downarrow})|\Psi_0\rangle$ (for $C^{(c)}$), and $(n_{i\uparrow} - n_{i\downarrow})|\Psi_0\rangle$ (for $C^{(s)}$). Second, we identify the excitation content of the states by inspecting the corresponding motifs where the spinons, holons, and antiholons can easily be identified (see Table I). We empirically find the following selection rules that the holon (v_h), spinon (v_s), antiholon ($v_{\bar{h}}$), spin wave (v_s^0), and sound (v_c^0) velocities always satisfy: (i) $v_c^0 < v_s^0$ (i.e., spin-charge separation), (ii) $v_c^0 \le |v_h|(|v_s|) \le v_s^0$, (iii) $|v_{\bar{h}}| \le v_c^0$, and (iv) for a given spinon-holon pair (s_R, h_R), $|v_{s_R}| \ge |v_{h_R}|$. These rules together with the results in Table I allow us to plot the regions of compact support as shown in Figs. 1–3.

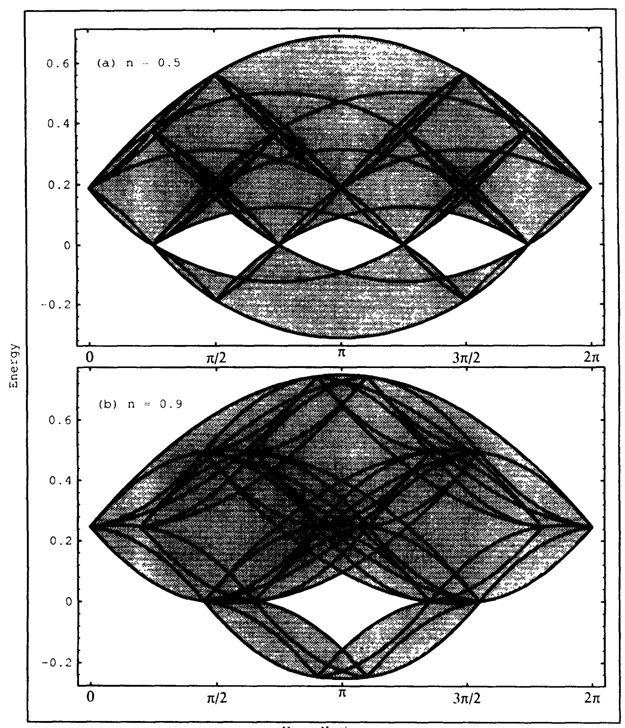

FIG. 1. *Compact support* for the one-particle Green's function. The momentum is in units of π and the excitation energy E in π^2/t. The contributing elementary excitations to this region are $(h_L, s_L) + \bar{h} + 2(h_R, s_R)$ for the positive energy part (i.e., $c_{i\sigma}|\Psi_0\rangle$) and $(s_L, h_L) + \bar{h}$ for the negative part (i.e., $c_{i\sigma}^\dagger|\Psi_0\rangle$). Their mirror states (i.e., L and R exchanged) also contribute. The four momenta at which $E = 0$ is allowed are k_F, $2\pi - 3k_F$, $3k_F$, and $2\pi - k_F$ where $k_F = \pi\bar{n}/2$.

Figure 1 shows the region of support for the one-particle Green's function where the states $c_{i\sigma}|\Psi_0\rangle$ ($c_{i\sigma}^\dagger|\Psi_0\rangle$) propagate in time with positive (negative) energy with respect to the ground state. The spectral functions should be nonanalytic along all the solid lines where the elementary excitations either "touch" the boundaries or the other elementary excitations. When the antiholons are suppressed (i.e., near half filling), the holon is accompanied either by a spinon or by three spinons in $S = 1/2$ state. At $3k_F$ ($2\pi - 3k_F$), where $k_F = \pi\bar{n}/2$, the left (right) moving spinon is missing from the state $c_{i\sigma}^\dagger|\Psi_0\rangle$, since the charge conservation prevents more than one holon in the presence of one antiholon. Of course, if two antiholons were allowed then states of the type $(s_L, h_L) + 2\bar{h} + 2(s_R, h_R)$ would contribute. Our numerical study indicates that states with two antiholons do not contribute. In fact, the observed states listed in Table I are the simplest possible states satisfying the charge (spin) conservation with at most one antiholon.

In Fig. 2, only holon-antiholon branches are present at $4k_F$ ($2\pi - 4k_F$), while the spinon-holon branches show up at $2k_F$ ($2\pi - 2k_F$). In Fig. 3 we find that the pure spinon excitations are possible only if they both belong to

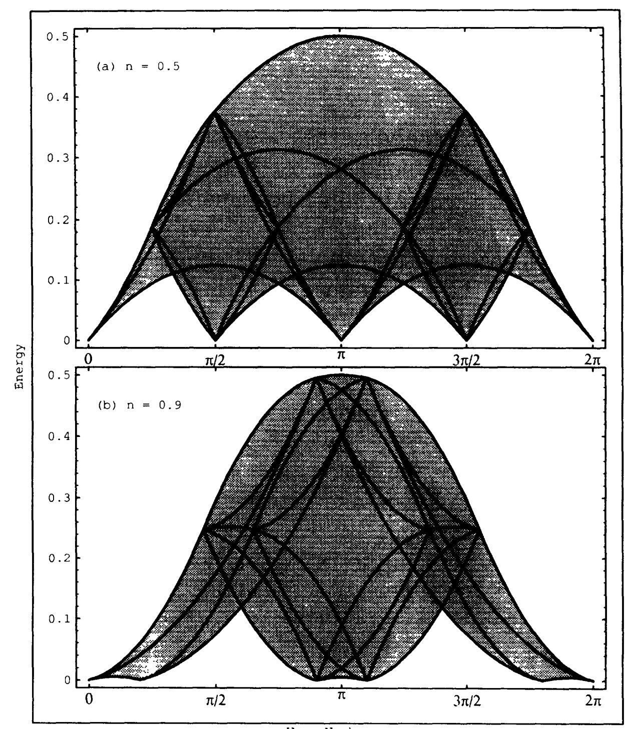

FIG. 2 *Compact support* for the density-density correlation function. $(s_L, h_L) + \bar{h} + (s_R, h_R)$, $\bar{h} + 2h_R$ and their mirror states contribute. $E = 0$ is allowed at $0(2\pi)$, $2k_F$, $2\pi - 4k_F$, $4k_F$, $2\pi - 2k_F$. Only holon-antiholon branches are present at $4k_F$ ($2\pi - 4k_F$) indicating that $4k_F$ is the holon Fermi point.

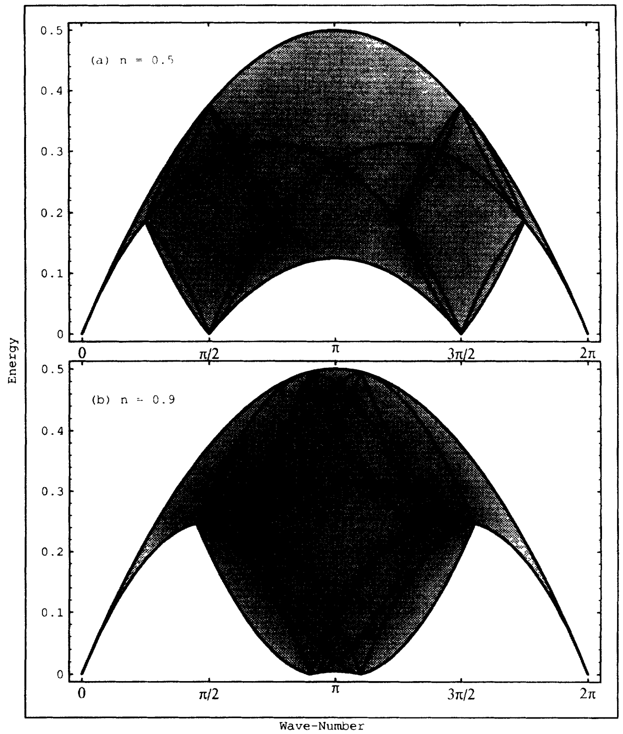

FIG. 3. *Compact support* for the spin-spin correlation function. $(s_L, h_L) + \bar{h} + (s_R, h_R)$, $2s_L$, and their mirror states contribute. $E = 0$ allowed at $0(2\pi)$, $2k_F$, $2\pi - 2k_F$. This indicates that $2k_F$ is the spinon Fermi point.

the same sector, otherwise they are accompanied by two holons and an antiholon. The excitation content we find for $S_i^z [= (n_{i\uparrow} - n_{i\downarrow})/2]$ should be same for $S_i^{\pm}$, since the ground state is a spin singlet. As $\bar{n} \rightarrow 0$ we recover the two spinon spectrum for the $S = 1/2$ spin chain.

Finally, we have examined how the ISE results for the charge of the elementary excitations change if we interpolate between the ISE and NNE t-J models, which are, respectively, the $\gamma = 0$ and $\gamma = \infty$ limits of the integrable family of *hyperbolic* models with exchange $\propto 1/\sinh^2 \gamma(i - j)$ [8]. Away from the ISE limit, the charge carried by the holon and antiholon excitations vary with their velocity, and become equal in magnitude (and opposite in sign) as the velocities approach the sound velocity v_c^0. In the ISE limit, however, the holon charge ($|v| > v_c^0$) is always $-e$, and the antiholon charge ($|v| < v_c^0$) is always $+2e$. The "dressed charge" carried by the excitations can be calculated using the asymptotic Bethe ansatz [18].

In conclusion, we have devised simple rules for constructing the motifs for the excited states of the 1D ISE t-J model and identified the exact excitation content of the intermediate states for the one-particle Green's function, the charge density-density and spin-spin correlation functions. We believe that this model is in the same universality class as the NNE model, and that the most relevant states for the ground-state correlation functions of the NNE model are also given by Table I. Finally, the presence of spinons, holons, and antiholons in two-dimensional models and their role in the high T_c superconductivity is an amusing possibility.

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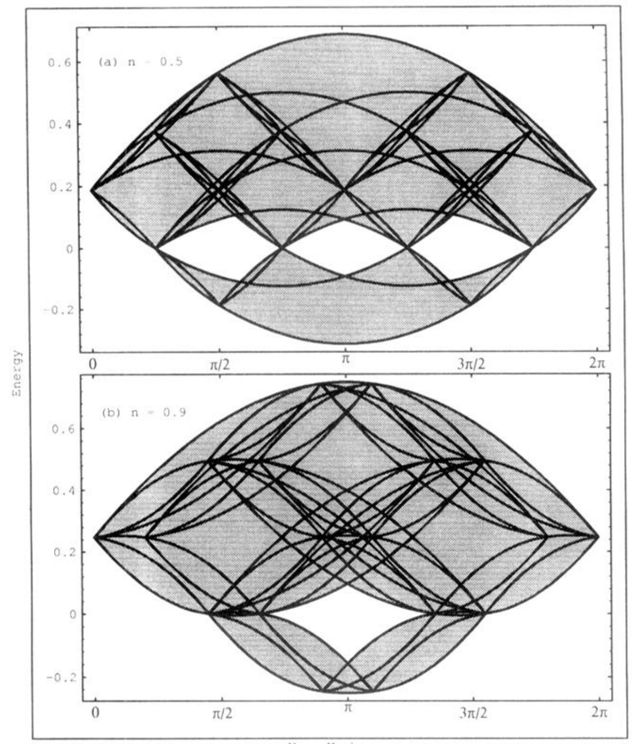

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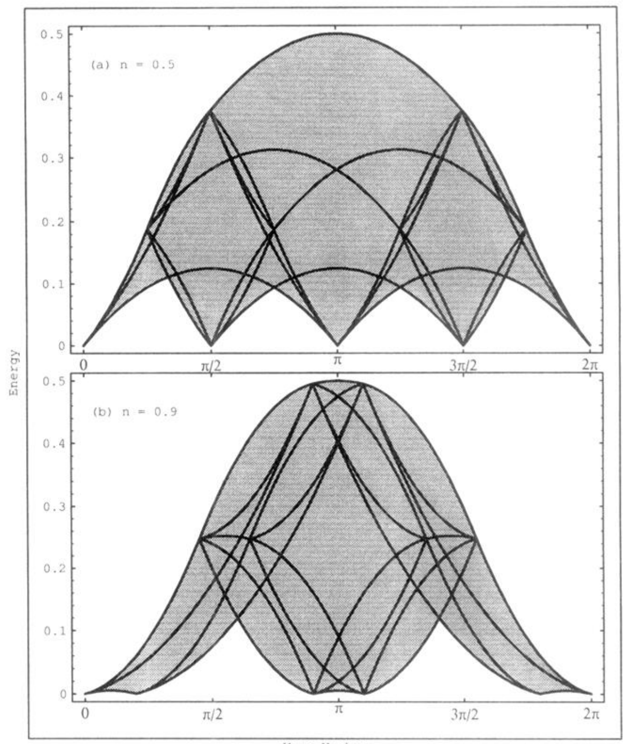
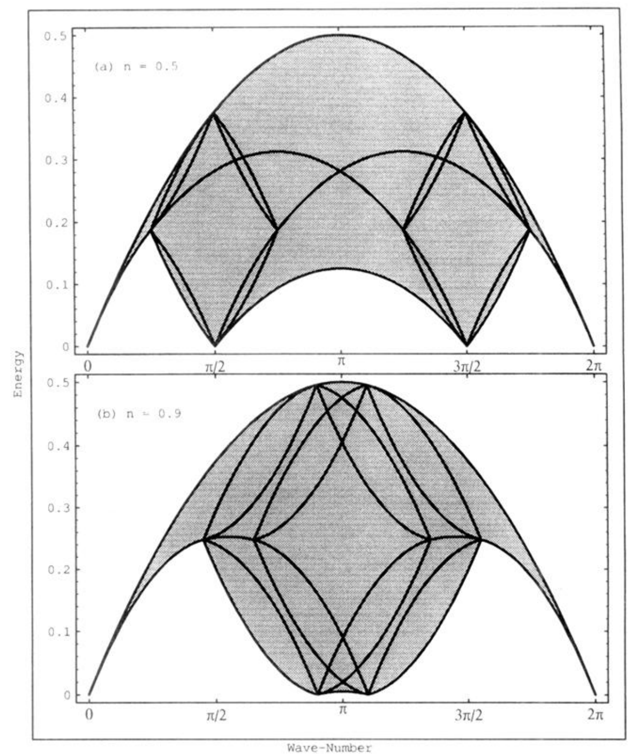


FIG. 2 *Compact support* for the density-density correlation function. $(s_L, h_L) + \bar{h} + (s_R, h_R)$, $\bar{h} + 2h_R$ and their mirror states contribute. $E = 0$ is allowed at $0(2\pi)$, $2k_F$, $2\pi - 4k_F$, $4k_F$, $2\pi - 2k_F$. Only holon-antiholon branches are present at $4k_F$ $(2\pi - 4k_F)$ indicating that $4k_F$ is the holon Fermi point.



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