Bubble Breakup in Two-Dimensional Stokes Flow

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A new class of exact solutions is reported for an evolving bubble in a two-dimensional slow viscous flow. It is observed that for an expanding bubble the interface grows smoother with time, whereas the contracting-bubble solutions display a tendency to form sharp corners ("near cusps") for small values of surface tension. In the latter case, we also obtain analytic solutions that describe bubble breakup: For a large class of initial shapes, the interface will eventually develop a thin "neck" whose width goes to zero before the bubble is completely removed from the liquid.

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The dynamical process whereby a single mass of fluid breaks into two or more pieces has recently become the subject of intensive theoretical [1–4] and experimental [5] research. One of the motivations for these studies has been the realization that such a phenomenon provides a simple example of singularity formation in a hydrodynamic system [6]. The analogous problem of bubble breakup has received considerably less attention. It should be noted nevertheless that there exists extensive literature on the deformation and burst of small drops and bubbles in a shear flow [7]. In many of these studies, however, the term "drop breakup" generally refers to the nonexistence of a steady solution when the applied shear strength exceeds some critical value; fewer works have considered the actual fragmentation of the drop or bubble [8].

In this Letter, we report what we know to be the first instance where analytic solutions describing the bubble shape up to the breakup point have been found. More specifically, we present below a general class of exact solutions for an evolving bubble in a two-dimensional viscous flow in which inertial effects are neglected (Stokes flow). Our solutions are given in terms of a conformal mapping and can describe both expanding and contracting bubbles in an otherwise quiescent flow. In the case of growing bubbles, the solutions have a simple behavior in the sense that for any given initial shape the bubble will asymptote an expanding circle. The case of contracting bubbles, on the other hand, displays an array of rather interesting phenomena. First, if surface tension is neglected, the solutions will in general develop a cusplike singularity before the bubble fluid ("air") is totally extracted. The inclusion of a small surface tension, as expected, prevents the occurrence of actual cusps and leads to the formation of "narrow structures" (i.e., near cusps whose "radius of curvature" vanishes exponentially with the surface tension parameter) similar to those observed in a viscous drop with suction [9]. As the bubble contracts, two scenarios are possible: (i) If initially the bubble either has an elliptical shape or possesses nth-fold symmetry (i.e., invariance under rotation by $2\pi/n$, for $n > 2$), then the bubble area will shrink to zero; but (ii) with neither symmetry initially present, the interface will develop a thin "neck" whose width becomes zero before the air is completely removed.

Physically, the second scenario above means that the contracting bubble (e.g., a dissolving gas bubble) would eventually break up into two or more bubbles before disappearing completely. Since our solutions break down at the time when the two sides of the interface "touch" each other, we are unable to follow the dynamics of the newborn bubbles. We emphasize, however, that contrary to the cases of singularity formation mentioned in the first paragraph no physical quantity blows up as the bubble approaches "breakup." Thus we refer to this process as a "topological singularity" caused by the loss of univalence of the conformal mapping.

It has recently been noted [9] that there are interesting similarities between free-boundary problems in Stokes flow and analogous problems occurring in the widely studied Hele-Shaw cell [10]. For example, in both systems, solutions for a viscous drop with suction in the absence of surface tension will generically develop a cusp singularity before the fluid is completely removed [9]. Our results, however, reveal a major distinction between these two systems. Cusp formation in a Hele-Shaw cell (at zero surface tension) occurs in the unstable displacement of a more viscous fluid by a lesser one [11]. In Stokes flows, on the other hand, such a displacement has no necessary bearing on cusp formation—the expanding bubble does not lead to cusps, whereas rather unexpectedly from the viewpoint of the Hele-Shaw analogy, bubble contraction does so (for zero surface tension).

The formulation (in terms of conformal mapping) of the problem of an inviscid bubble placed in a two-dimensional Stokes flow parallels that of a 2D viscous drop, for which exact solutions have recently been reported [12,13]. We first recall that the problem of 2D Stokes flow can be conveniently formulated in terms of a stream function $\psi(x, y)$, defined as

$$u_1 = \frac{\partial \psi}{\partial y}, \qquad u_2 = -\frac{\partial \psi}{\partial x}, \tag{1}$$

where u_1 and u_2 are the x and y components of the fluid velocity, respectively. Here ψ obeys the biharmonic equation [14]

$$\nabla^4\psi = 0, \tag{2}$$

so that we can use the Goursat representation for biharmonic functions to write [15]

$$\psi = \mathrm{Im}[\bar{z}f(z) + g(z)], \tag{3}$$

where $f(z)$ and $g(z)$ are analytic functions of $z = x + iy$ in the fluid region. One also has the following identities:

$$\frac{p}{\mu} - i\omega = 4f'(z), \tag{4}$$

$$u_1 + iu_2 = -f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}), \tag{5}$$

where p is the pressure, ω is the vorticity, and μ is the viscosity. Here prime indicates derivative and $\bar{f}$ denotes the so-called conjugate function $\bar{f}(z) = \overline{f(\bar{z})}$ (and similarly for $\bar{g}$).

On the bubble surface, we must also satisfy (i) continuity of the shear stress and (ii) the requirement that the jump in the normal stress across the interface equals the product of the surface tension σ times the curvature κ. These two conditions can be written as one complex equation involving the functions f and g as follows [16]:

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -\frac{i\sigma}{2\mu} z_s, \tag{6}$$

where s is the arclength traversed in the clockwise direction and the subscript denotes partial derivative.

Finally, to completely specify the problem we need to prescribe appropriate boundary conditions at infinity. For a bubble whose area is changing at a rate m, it can be shown that [16]

$$g'(z) \sim \frac{m}{2\pi z} + O(1/z^2), \qquad \text{as } |z| \longrightarrow \infty, \tag{7}$$

Notice that m can in principle be any arbitrary function of time. Henceforth, we will take m to be a negative constant ($m < 0$) to focus on the case of contracting bubbles. We look for solutions with vorticity vanishing at infinity so that Eq. (4) implies

$$f(z) \sim \frac{p_\infty}{4\mu} z + O(1/z), \qquad \text{as } |z| \to \infty, \tag{8}$$

where $p_\infty(t)$ is the pressure at $z = \infty$. Here, however, p_∞ cannot be specified *a priori* but rather is determined *a posteriori* once the bubble shape is known [16].

Next we introduce the conformal mapping $z(\zeta, t)$ that maps the interior of the unit circle in the ζ complex plane to the fluid region (i.e., the exterior of the bubble) in the z plane, such that the $\zeta = 0$ corresponds to the point $z = \infty$. We thus write

$$z(\zeta, t) = \frac{a(t)}{\zeta} + h(\zeta, t), \tag{9}$$

where $a(t)$ is chosen to be real and negative and $h(\zeta, t)$ is analytic in the interior of the unit circle ($|\zeta| < 1$). The kinematic condition that on the interface the normal component of the fluid velocity equals the normal component of the interface velocity yields the following boundary condition on the unit circle ($|\zeta| = 1$):

$$\mathrm{Re}\left[\frac{z_t + 2F(\zeta, t)}{\zeta z_\zeta}\right] = \frac{\tau}{2|z_\zeta|}, \tag{10}$$

where $F(\zeta, t) = f(z(\zeta, t), t)$ and $\tau = \sigma/\mu$. We notice that the quantity within square brackets is an analytic function of ζ in $|\zeta| < 1$. (Note that the simple poles at $\zeta = 0$ in both the numerator and denominator cancel out.) It thus follows from the Poisson formula [15] that for $|\zeta| < 1$

$$z_t + 2F(\zeta, t) = \tau\zeta I(\zeta, t) z_\zeta, \tag{11}$$

where

$$I(\zeta, t) = \frac{1}{4\pi i}\oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'}\left[\frac{\zeta + \zeta'}{\zeta' - \zeta}\right]\frac{1}{|z_\zeta(\zeta', t)|}. \tag{12}$$

We now present our exact solutions to the problem. We first note that it can be shown [16] that the nature of the singularities (including that at $\zeta = \infty$) of the function $h(\zeta, t)$ is preserved by the dynamics given in Eq. (11) (or rather its analytical continuation to $|\zeta| > 1$). This implies, in particular, that if h is initially a polynomial of degree N, then it must remain so for all times for which the solution exists [17]. We thus seek solutions of the form

$$h(\zeta, t) = \sum_{j=1}^{N} b_j(t)\zeta^j, \tag{13}$$

where b_j are real coefficients. The problem now consists of finding a system of ordinary differential equations (ODE's) for the coefficients $a(t)$ and $b_j(t)$. This can be achieved as follows.

We first define the function $G(\zeta, t) = g'(z(\zeta, t), t)$, which is analytic in $|\zeta| < 1$. Moreover, in view of Eq. (9), the asymptotic condition (7) implies that

$$G(\zeta, t) = \frac{m}{2\pi a}\zeta + O(\zeta^2), \qquad \text{as } \zeta \longrightarrow 0. \tag{14}$$

Now consider the boundary condition (6) recast in the ζ variable. Using Eq. (11) to eliminate $F(\zeta, t)$ from Eq. (6), we can then write $G(\zeta, t)$ in terms of $z(\zeta, t)$. Once this is accomplished, the requirement that $G(\zeta, t)$ behaves as in Eq. (14) will immediately produce a set of ODE's for the coefficients of the mapping function $z(\zeta, t)$.

Here we take up this program for the particular case in which $z(\zeta, t)$ is given by

$$z(\zeta, t) = \frac{a(t)}{\zeta} + b(t)\zeta + c(t)\zeta^3, \tag{15}$$

where $b(t)$ and $c(t)$ [along with $a(t)$] are real. Carrying out the procedure described in the preceding paragraph, we obtain the following system of ODE's [16]:

$$\dot{X} = -4\tau I_0 X, \tag{16}$$

$$\dot{Y} = -2\tau I_2 X - 2\tau I_0 Y, \tag{17}$$

$$\dot{Z} = \frac{m}{\pi}, \tag{18}$$

where $X = ac$, $Y = b(a - c)$, $Z = a^2 - b^2 - 3c^2$, $I_0 = I(0, t)$, and $I_2 = \frac{1}{2}I_{\zeta\zeta}(0, t)$. Notice that I_0 and I_2 depend on a, b, and c in a manner that can be calculated from Eq. (12). Now noting that the area A enclosed by the curve obtained as the image of the unit circle under the mapping (15) is given by $A = \pi(a^2 - b^2 - 3c^2)$, we immediately see that Eq. (18) is equivalent to $\dot{A} = m$. After solving this equation explicitly (for constant m), we numerically integrate Eqs. (16) and (17) for specified initial data. Before discussing the general case above, however, it is instructive to examine first the following two particular cases: (i) elliptical bubbles, for which $c = 0$, and (ii) bubbles with fourfold symmetry, in which case $b = 0$.

Elliptical bubbles.—Setting $c = 0$ in Eqs. (17) and (18) and solving for $\dot{b}$ yields

$$\dot{b} = -\frac{b}{a^2 + b^2}\left[2\tau a^2 I_0 + \frac{m}{2\pi}\right], \tag{19}$$

plus the area constraint: $a^2 - b^2 = [A(0) + mt]/\pi$, where $A(0)$ is the area of the original bubble. (Here for definiteness we assume $b < 0$ so that the ellipse major axis is aligned with the x direction.) This ODE is particularly simple to analyze if surface tension is taken to be zero ($\tau = 0$). In this case, one immediately sees that b increases and a decreases in magnitude monotonically with time (recall that $m < 0$). Since the bubble area must vanish at $t = t_f = A(0)/|m|$, we must then have $a(t_f) = b(t_f) \neq 0$. In other words, the final stage of the bubble is a *slit* of length $2|a(t_f)|$ along the x axis. In the case of nonzero surface tension ($\tau \neq 0$), an asymptotic analysis of Eq. (19) in the limit of vanishing area shows [16] that the bubble will also shrink to a slit whose size decreases with τ.

Fourfold symmetric bubbles.—In this case we set $b = 0$ and solve Eqs. (16) and (18) for $\dot{c}$:

$$\dot{c} = -\frac{c}{a^2 + 3c^2}\left[4\tau a^2 I_0 + \frac{m}{2\pi}\right], \tag{20}$$

plus the area condition $a^2 - 3c^2 = [A(0) + mt)]/\pi$. (Here for definiteness we assume $c > 0$ initially.) A qualitative analysis of this ODE can also be easily performed. For instance, in the case of zero surface tension ($\tau = 0$), one can readily verify that c increases and a decreases (in magnitude) monotonically with time. This implies, in turn, that there will be a time $t = t_c < t_f$ for which $a = -3c \neq 0$. In other words, at $t = t_c$ the critical points $\zeta_0 = e^{i\pi/4}(-a/3c)^{1/4}$ of the conformal mapping [i.e., the zeros of $z_\zeta(\zeta, t)$] impinge on the unit circle, thus leading to the formation of cusps on the interface and the subsequent breakdown of the solution.

The inclusion of surface tension is expected on general grounds to prevent the formation of actual cusps. That this is indeed the case here can be seen easily by examining the right-hand side of Eq. (20). Initially, as c increases, the zeros ζ_0 march in toward the unit circle. As they approach the unit circle, however, the quantity I_0 will grow bigger [recall that $I_0 = I(0, t)$ is always positive and diverges if a zero lies on the unit circle; see Eq. (12)], so as to slow down their motion, thus causing the formation of "narrow structures" (near cusps) on the bubble surface. In Fig. 1 we show a sequence of interface shapes, up to the formation of the near cusps, for $m = -1$ and $\tau = 0.1$. A more detailed asymptotics analysis reveals [16] that, once formed, these near cusps persist until the final time: the bubbles will subsequently shrink to a point through a succession of geometrically similar shapes.

In the example above, we have seen in some detail how surface tension effects prevent the formation of cusplike singularities and guarantee that the air will be completely removed from the liquid. Recall also that for elliptical bubbles, total removal of air is always attained (although the bubble shrinks to a slit rather than a point). The situation changes considerably, however, when both b and c are initially nonzero. In such cases, a new phenomenon occurs: The bubble will develop a "thin neck" whose width will go to zero at a time $t = t_b < t_f$. This is illustrated in Fig. 2, where we show a sequence of interface shapes leading to the bubble breakup. Beyond the touching time t_b, the solution ceases to be physically meaningful since the mapping function $z(\zeta, t)$ is no longer one to one. We note, however, that the mathematical solutions can be continued for $t > t_b$, in which case the two sides of the interface would simply "cross" each other. Thus, while there is a "topological singularity," the mapping function has no singularity at the pinching point and the neck width goes to zero linearly in time.

We have thus seen that due to the breakup mechanism, our solutions above will always "fail" to remove the air completely from the fluid so long as $bc \neq 0$ initially. More generally, this will always be the case whenever the initial shape possesses neither elliptical nor nth-fold symmetry [16]. It should be mentioned nevertheless that the larger the surface tension (or slower the suction rate)

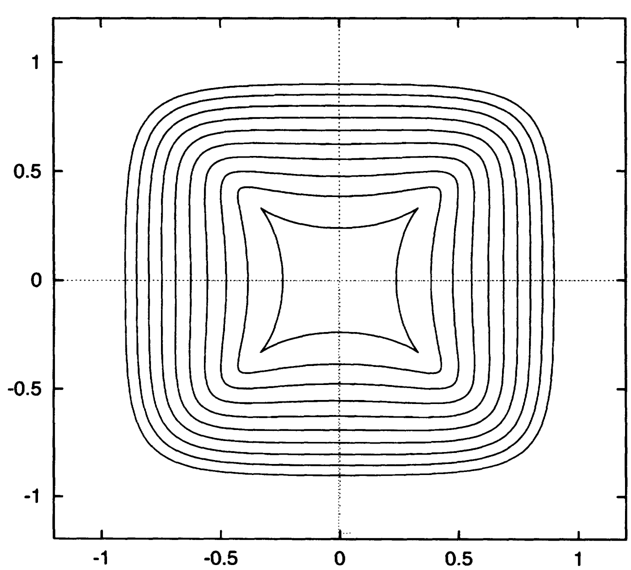

FIG. 1. The evolution of a fourfold symmetric bubble for $m = -1$ and $\tau = 0.1$. Note the formation of "near cusps" on the innermost interface.

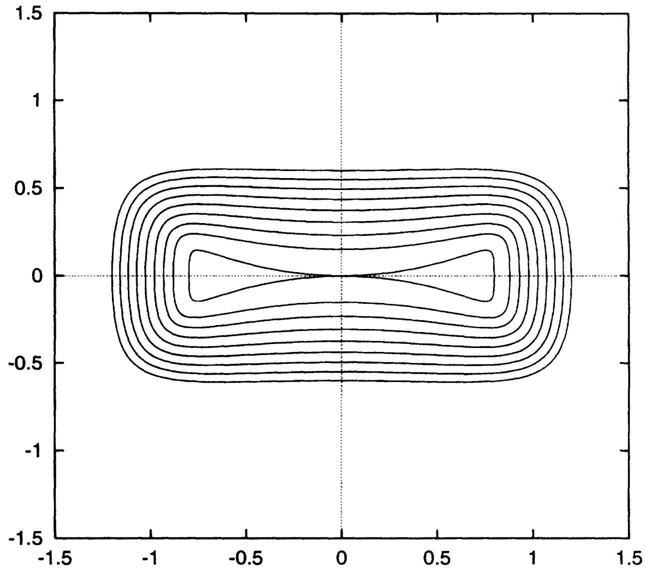

FIG. 2. The evolution of an "asymmetric" bubble leading to breakup. Parameters are as in Fig. 1.

the more effective this partial removal will be, that is, the smaller the bubble area at the time of breakup. We also note that higher order polynomials allow for breakup into more than two bubbles [16].

As a concluding remark, we would like to add that we have also studied the case in which m is taken to be proportional to the bubble perimeter, as a possible model for a gas bubble dissolving into the liquid. Here the results obtained were quite analogous [16] to those reported above for constant m. In particular, bubble breakup was observed for a large class of initial shapes. The dynamics beyond breakup (where presumably conformal mapping for multiply connected domain might be useful), as well as the effect of a nonzero viscosity ratio on the breakup, remain to be analyzed. (In the latter case, the interior flow can no longer be neglected and singularities in the gradients of fluid velocity inside the bubble are expected to develop at the pinchoff point.) While our results above are for a purely 2D Stokes bubble, which might not be experimentally feasible, from the similarities of the equations one might expect similar qualitative behavior for 3D axisymmetric bubble, though the analogy is likely to break down near pinching.

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