Conformal Charge and Exact Exponents in the $n = 2$ Fully Packed Loop Model

In a recent Letter Blöte and Nienhuis [1] study a fully packed loop (FPL) model on the honeycomb lattice for different values n of the loop fugacities. Here we show that the $n = 2$ model maps onto a three-coloring model [2] with additional symmetry; by mapping this model onto a solid-on-solid (SOS) model in *four* dimensions we confirm the *exact* values of the conformal charge, temperature, and magnetic dimensions found numerically in [1].

In Baxter's three-coloring model [2], every bond of the honeycomb lattice is assigned a color A, B, or C, so that colors A, B, and C meet at each vertex, and all such configurations are given equal weight. The A- and B-colored bonds form an FPL configuration, and since each loop can be colored either $(ABAB\dots)$ or $(BABA\dots)$ the corresponding fugacity is $n = 2$.

Each three-colored configuration can in turn be mapped onto an SOS model configuration, in which a *two component* height is defined in each hexagon of the honeycomb lattice [3]; the TISOS height of [1] is one of the components. The change in height when going from one hexagon to the neighboring one is given by $\vec{A}, \vec{B}$, or $\vec{C}$ depending on the color of the bond that is crossed. $\vec{A}$, $\vec{B}$, and $\vec{C}$ point to the vertices of an equilateral triangle. Thus, the $n = 2$ FPL model is equivalent to a freely fluctuating two-dimensional interface in *four* dimensions. Assuming that this interface is *rough*, the free energy of the SOS model, in terms of the *coarse grained* height $\vec{h}(x,y) = (h_1(x,y), h_2(x,y))$, has the form

$$f = \frac{K}{2} \int dx\, dy (|\vec{\nabla} h_1|^2 + |\vec{\nabla} h_2|^2). \qquad (1)$$

The two Gaussian degrees of freedom in Eq. (1) imply that the conformal charge $c = 2$.

Any operator $\Phi(x,y)$ constructed from the colors is a periodic function of $\vec{h}(x,y)$. The periods form a triangular lattice with lattice constant $|\vec{b}_1| = \sqrt{3}|\vec{A}|$, which we call the "direct lattice" $\mathcal{D}$. If this operator varies as $\Phi_{\vec{G}} = e^{i\vec{G}\cdot\vec{h}}$, where $\vec{G}$ is in the reciprocal lattice of $\mathcal{D}$, then $\Phi_{\vec{G}}$ has an algebraically decaying autocorrelation function [4], with dimension

$$2X(\vec{G}) = \frac{1}{2\pi} \frac{|\vec{G}|^2}{K}. \qquad (2)$$

The operator which restricts the *microscopic* heights to discrete values has a periodicity given by the second shortest reciprocal lattice vector $\vec{G}_2$. As shown by [3], Baxter's exact solution [2] implies $X(\vec{G}_2) = 2$. This means the SOS model is (marginally) rough, justifying the form of Eq. (1). The marginality means the $n = 2$ FPL model is *at* its roughening transition; increasing n corresponds to increasing the stiffness K and would drive the SOS model into the flat phase.

In order to calculate the magnetic and temperature dimensions, we must consider defects in the three colorings. In the SOS language these defects correspond to vortices with Burgers' vectors $\vec{b}$ taking values in $\mathcal{D}$. The various vortex-antivortex correlation functions decay algebraically with exponents of the form [4]

$$2X_v(\vec{b}) = \frac{1}{2\pi} K|\vec{b}|^2. \qquad (3)$$

The magnetic dimension governs the probability that two points in the honeycomb lattice lie on the same loop [1]. From an analysis similar to that of Nienhuis [4], we find that this is equivalent to the vortex-antivortex correlation function with Burgers vector $\vec{b}_1$, the shortest vector in $\mathcal{D}$. Thus, from Eq. (3), the magnetic dimension is $X_v(\vec{b}_1) = 1/2$; the value of K was obtained from $X(\vec{G}_2) = 2$.

The temperature dimension is associated with x^{-1} in [1]; this is the fugacity of vertices uncovered by a loop (so $x^{-1} \to 0$ in the FPL model). In the three-coloring model, an uncovered vertex becomes a defect with the same color on all three of its surrounding bonds; in the SOS language this becomes a vortex with Burgers' vector $\vec{b}_2$, the second shortest vector in $\mathcal{D}$. Hence, using Eq. (3), the temperature dimension is $X_v(\vec{b}_2) = 3/2$.

The color correlations are governed by the operator $\Phi_{\vec{G}_1}$, where $\vec{G}_1$ is the shortest reciprocal lattice vector, so that $X(\vec{G}_1) = 2/3$ [3]. When we consider the transfer matrix for this model wrapped on a torus, we expect its second largest eigenvalue to be associated with $\Phi_{\vec{G}_1}$. It is unclear to us why this eigenvalue does not appear, as shown by the data of Ref. [1].

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