Exclusion Statistics: Low-Temperature Properties, Fluctuations, Duality, and Applications

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(Received 9 May 1994)

We derive some physical properties of ideal assemblies of identical particles obeying generalized exclusion statistics. We discuss fluctuations and in this connection point out a fundamental contrast to conventional quantum statistics. We demonstrate a duality relating the distribution of particles at statistics g to the distribution of holes at statistics $1/g$. We suggest an application to Mott insulators.

PACS numbers: 73.40.Hm, 05.30.–d, 71.30.+h

Several years ago, Haldane [1] introduced the notion of generalized or fractional exclusion statistics, interpolating between bosonic and fermionic statistics. Motivated by the properties of quasiparticles in the fractional quantum Hall effect and in one-dimensional inverse-square exchange spin chains, he defined the statistics g of a particle by

$$g = -\frac{d_{N+\Delta N} - d_N}{\Delta N}, \tag{1}$$

where N is the number of particles and d_N is the dimension of the one-particle Hilbert space obtained by holding the coordinates of $N - 1$ particles fixed. Since any number of bosons can occupy a given state, $d_{N+\Delta N} = d_N$, and hence $g = 0$. By contrast, the Pauli exclusion principle implies that $g = 1$ for fermions. Particles with intermediate statistics—Laughlin quasiparticles with $g = 1/m$ and 1D spinons with $g = \frac{1}{2}$ were the two examples given in [1]—satisfy a generalized exclusion principle. Recently Wu [2] has discussed the statistical mechanics of particles obeying a generalized exclusion principle *locally in phase space* (see below). Henceforth we shall call such particles g-ons. Ouvry [3] had previously discussed a related statistical distribution in the context of anyons in a magnetic field. Bernard and Wu [4] have shown that excitations in the Calogero-Sutherland model obey this statistical mechanics. (For the exact interpretation of this statement, see below.)

An earlier form of exotic statistics—anyons [5,6]—has proved very influential in the study of two-dimensional systems. Anyons are particles whose wave functions acquire an arbitrary phase $e^{i\theta}$ when two of them are braided. Unlike fractional exclusion statistics particles, anyons are special to two dimensions since in higher dimensions two exchanges can be continuously deformed to no exchange. At first glance, exclusion statistics seem to have little to do with the braiding properties of particle trajectories which are the starting point for anyons. However, in a recent important paper, Murthy and Shankar [7] showed that anyons *do* satisfy a generalized exclusion principle (contrary to Haldane [1]). They did this by relating the exclusion statistics parameter g to the high-temperature limit of the second virial coefficient, which is nontrivial for an anyon gas [8]. The linchpin of their argument is that in a theory with a high-energy cutoff (e.g., any continuum model), the transmutation of statistics by attaching flux tubes [6] will generally push some states beyond the cutoff, thereby reducing the Hilbert space dimension. This generates a fractional exclusion statistics that persists even as the cutoff is taken to infinity.

We now briefly recall the framework of g-on statistical mechanics [1,2]. Following [9], we imagine dividing the one-particle states into a large number of cells with $k \gg 1$ states in each cell and then count the number of configurations with n_i particles in the ith cell. An elementary combinatorial argument gives

$$e^S = \prod_i \frac{(d_{n_i} + n_i - 1)!}{n_i!\,(d_{n_i} - 1)!}, \tag{2}$$

where d_{n_i} is the dimension of the one-particle Hilbert space in the ith cell with the coordinates of $n_i - 1$ of the particles held constant. If one can apply the definition (1) *locally in phase space*—a big assumption that we tentatively adopt as the working definition of g-ons, but will need to discuss critically and refine shortly—then

$$d_{n_i} = k - g(n_i - 1). \tag{3}$$

Hence we must minimize the quantity

$$\begin{aligned}\overline{S} = \sum_i &[k + (1-g)(n_i - 1)]\ln[k + (1-g)(n_i - 1)] \\ &- n_i \ln n_i - [k - g(n_i - 1) - 1] \\ &\times \ln[k - g(n_i - 1) - 1] - \beta\epsilon_i n_i + \beta\mu n_i\end{aligned} \tag{4}$$

with respect to the occupation numbers, n_i; β and $\beta\mu$ are the Lagrange multipliers which enforce the constraints of fixed energy and particle number. Differentiating with respect to n_i, we obtain [neglecting terms of $O(1)$ which are negligible compared to ln k],

$$\begin{aligned}\frac{\partial \overline{S}}{\partial n_i} = (1 - g)\ln[k + (1-g)n_i] - \ln n_i + g\ln(k - gn_i) \\ -\beta(\epsilon_i - \mu).\end{aligned} \tag{5}$$

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Upon exponentiation this yields

$$\overline{n}_i e^{\beta(\epsilon_i - \mu)} = (1 - g\overline{n}_i)^g [1 + (1 - g)\overline{n}_i]^{1-g}, \quad (6)$$

where we have written $\overline{n}_i = n_i/k$ for short. This is the fundamental equation that implicitly defines the distribution function for g-ons. In general it can only be solved numerically, but for special cases including $g = 0, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4$ it can be solved analytically. The $g = \frac{1}{2}$ distribution function is

$$(\overline{n}_i)_{g=1/2} = \frac{2}{(1 + 4e^{2\beta(\epsilon_i - \mu)})^{1/2}}. \quad (7)$$

Low-temperature properties.—At $T = 0$ the distribution function vanishes for $\epsilon > \mu$ and takes the value 2 at $\epsilon < \mu$. More generally, one can show by inspection of (6) that at $T = 0$

$$(\overline{n}_i)_g = \begin{cases} g^{-1}, & \text{if } \epsilon < \mu, \\ 0, & \text{if } \epsilon > \mu. \end{cases} \quad (8)$$

It is quite striking that at $T = 0$ particles of general exclusion statistics exhibit a "Fermi" surface. This fact dictates the low-temperature thermodynamics of systems of these particles when the particle number is conserved.

We can develop an expansion in powers of the temperature—analogous to the Sommerfeld expansion for fermions—for the thermodynamic functions of g-ons.

Let the single-particle energies be $\epsilon(p) = ap^n$. We find, up to exponentially small corrections,

$$\mu_0^{D/n} = \mu^{D/n}\left[1 + g(D/n)\sum_j \left(\frac{T}{\mu}\right)^{j+1} \binom{-1 + D/n}{j} C_j\right], \quad (9)$$

$$E = E_0\left(\frac{\mu}{\mu_0}\right)^{D/n+1}\left[1 + g(1 + D/n) \times \sum_j \left(\frac{T}{\mu}\right)^{j+1} \binom{D/n}{j} C_j\right], \quad (10)$$

where D is the spatial dimension, μ_0 and E_0 are the zero-temperature chemical potential and energy, respectively, and we have isolated the pure numbers

$$C_j = \int_0^\infty d\eta\, \eta^j \left[\overline{n}_g(\eta) + (-1)^j\left(\overline{n}_g(-\eta) - \frac{1}{g}\right)\right]. \quad (11)$$

In the special case $g = \frac{1}{2}$, C_0 can be evaluated analytically. Using the distribution function (7), one finds rather surprisingly

$$(C_0)_{g=1/2} = \int_0^\infty du \frac{1}{(1 + 4e^u)^{1/2}} + \int_{-\infty}^0 du \left(\frac{1}{(1 + 4e^u)^{1/2}} - 1\right)$$
$$= \frac{1}{2}\int_2^\infty \frac{dv}{v} \frac{1}{(1 + v^2)^{1/2}} + \frac{1}{2}\int_0^2 \frac{dv}{v}\left(\frac{1}{(1 + v^2)^{1/2}} - 1\right) = 0. \quad (12)$$

Hence, the $g = \frac{1}{2}$ specific heat to order T is simply

$$\left(\frac{C}{T}\right)_{g=1/2} = (1 + D/n)\frac{E_0 C_1}{\mu_0^2}. \quad (13)$$

C_1 may be evaluated numerically; $C_1 = 1.6449$. In the case of fermions $C_n = 0$ for all even n due to particle-hole symmetry, but this is not the case for $g = \frac{1}{2}$; specifically, $C_2 = 1.2021$. Remarkably, we find numerically that $C_0 = 0$ for $g = \frac{1}{3}, \frac{1}{4}$ as well. These results—together with a duality property to be demonstrated later which implies that $C_0 = 0$ for $g = 2, 3, 4$ as well—lead us to conjecture that $C_0 = 0$ for arbitrary g.

If the particle number is not conserved one has, immediately,

$$E = \sum_i \epsilon_i n_i = V\int \frac{d^D p}{(2\pi)^D}\epsilon(p)\overline{n}_g(\epsilon(p))$$
$$= \alpha V T^{D/n+1}\int d\eta\, \eta^{D/n}\overline{n}_g(\eta), \quad (14)$$

where $\alpha^{-1} = a^{D/n+1}\pi^{D/2}2^{D-1}\Gamma(D/2)$. For $\frac{1}{2}$-ons with $D/n = 1$, the integral in (14) takes the value 0.9870.

Fluctuations and a perspective on the assumptions.—Let us attempt to find the probabilities for various occupation numbers of a single state, for $g = \frac{1}{2}$. Defining $f(n)e^{-n\beta(\epsilon-\mu)}$ as the probability for n-fold occupancy, we derive from (7) the formal relation

$$\frac{\sum nf(n)e^{-n\beta(\epsilon-\mu)}}{\sum f(n)e^{-n\beta(\epsilon-\mu)}} = \frac{1}{[\frac{1}{4} + e^{2\beta(\epsilon-\mu)}]^{1/2}}. \quad (15)$$

Matching coefficients and normalizing $f(0) = 1$ we find $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{8}$, $f(4) = 0$, $f(5) = -\frac{1}{128}, \ldots$. Clearly something has gone awry here.

The mathematical problem is as follows. The dimension of the many-particle Hilbert space

$$W = \frac{[k + (1 - g)(N - 1)]!}{N![(k - 1) - g(N - 1)]!} \quad (16)$$

for N particles when the cell includes k states vanishes at $g = m/(N - 1)$ where $m = k, k + 1, \ldots, k + N - 1$. This gives the correct result for fermions, namely $W = 0$ if $N > k$ since $g = 1$ is then one of the zeros of W. However, W does not vanish for general g when

$gN > k$. Indeed, W can be negative when $gN > k$. (To see this, consider the simple case of $g = \frac{1}{2}$ for $N > 2k$ and N an even number.) To get a sensible result for the Hilbert space dimension W, we must stipulate that $W = 0$ if $gN > k$, which complicates the minimization of the entropy. If we take a large cell size k we can safely ignore this complication, since W is small for $gN > k$. For instance, in the case of semions, the error in ignoring the constraint $W_N = 0$ for $N > 2k$ is inversely proportional to the cell size, namely $|W_N| < 1/2k$ for $N > 2k$. Thus for g-ons it is important to keep the cell size large, in contrast to the Fermi and Bose cases where one could take $k = 1$ with impunity.

One must therefore exercise some care in using the distribution functions derived from (6). The cell size must not become too small in view of the preceding paragraph, but it must also not become too large, because if the energy spread within one cell becomes comparable to the temperature, then the notion of a characteristic energy for the cell becomes invalid. It is amusing that negative probabilities appear in this problem in a natural and meaningful way: It is necessary that negative probabilities appear in the description of small cells, if independent addition of many such is to generate the correct average occupancy for large cells.

Our mathematical problem reflects a fundamental implicit *physical* assumption in the derivation of g-on statistical mechanics. For bosons or fermions the fundamental assumption of symmetry or antisymmetry of the wave function holds rigorously and locally in momentum space. This is enough to allow one to derive the appropriate statistical distribution for an ideal gas locally in phase space. To derive g-on statistical mechanics as above one must assume that the generalized exclusion principle operates on states of nearby energy, and as we have seen one must also take a cell size not too small. Without attempting a rigorous discussion, we can identify qualitatively the physical circumstance under which these assumptions become plausible. It is that the effective interaction which reduces the Hilbert space dimensions should be essentially local in momentum space. Then one may apply the counting arguments to cells containing all the states in a small range of momenta: This will be a number of states proportional to the volume, all with essentially the same energy.

The known examples of g-ons have this character. Ideal g-on statistical mechanics operates in the Calogero-Sutherland models [4], which feature long-range interactions. (Actually there is an important subtlety here. The states of the Calogero-Sutherland model are exactly classified in terms of occupation numbers of momentum states obeying ideal g-on statistics. This is somewhat misleading, however, because the relation between these momenta and the energy of the state is complicated. In a gauge theory formulation of the model [10] this complication arises because the momenta used in the classification are canonical momenta, whereas the energy involves the kinetic momenta. The occupation numbers as a function of momentum are just those of the ideal g-on gas calculated with the energy-momentum relation of free particles. However, the actual energies of the corresponding one-particle states are complicated and are determined, according to the thermodynamic Bethe ansatz equations, by the condition that the g-on distribution as a function of the free particle energy is equal to the bosonic distribution as a function of the actual energy [4]. Nevertheless, the thermodynamic quantities are correctly calculated using the ideal g-on formulas.)

Anyon models [that is, (2 + 1)-dimensional systems with a Chern-Simons gauge field] feature interactions which are singular for nearby momenta, resulting in a shift of the allowed values of the relative angular momentum between two particles, $l \to l + \alpha$. Since two particles occupying the same state must have vanishing relative angular momentum, an anyon excludes its state from further occupation. This is certainly local exclusion, but the second condition is not satisfied: An anyon in any other state also has its relative angular momentum shifted. Hence anyons are not ideal g-ons, but *interacting* g-ons. Whether the ideal g-on statistical mechanics provides a useful first approximation in this case is a question needing further investigation. Hard-core bosons on a lattice have $g = 1$ according to Haldane's definition, but are far from being 1-ons according to our definition, and their behavior is poorly approximated by fermions: One expects them to Bose condense rather than to form a Fermi surface at low temperatures.

To conclude this discussion, let us finally display the first-order fluctuations concretely, for $g = \frac{1}{2}$. By differentiating (15) with respect to the chemical potential μ and rearranging terms, we find

$$\overline{(\Delta n)^2} = \bar{n}\left(1 - \frac{1}{4}\bar{n}^2\right). \tag{17}$$

Thus we find that semions have sub-Poissonian statistics, as do fermions $\overline{(\Delta n)^2} = \bar{n}(1 - \bar{n})$. In contrast, bosons are super-Poissonian, $\overline{(\Delta n)^2} = \bar{n}(1 + \bar{n})$.

Duality.—We alluded earlier to a duality property that relates statistics g and $1/g$. It is

$$1 - g\bar{n}_g[\beta(\epsilon - \mu)] = \frac{1}{g}\bar{n}_{1/g}\left(-\frac{\beta}{g}(\epsilon - \mu)\right) \tag{18}$$

and is not difficult to verify. This duality relates the distribution of holes in the g-on distribution (where full filling is at $\bar{n}_g = 1/g$) to that of $1/g$-ons at g times the temperature, or alternatively $1/g$ times the energy and chemical potential.

This duality is reminiscent of the one found in Chern-Simons models. There one describes anyons as charge $Q = q$ objects which acquire a proportional flux $\Phi = q/\mu$, where μ is the Chern-Simons coupling, and thereby have their statistics transmuted by $\Delta\theta/\pi = Q\Phi = q^2/\mu$. The fundamental flux tubes then have flux $\Phi = 1/q$, charge $Q = \mu/q^2$, and the inverse statistics. The thermal duality (18) is also reminiscent of the $g \rightarrow 1/g$ duality in the Calogero-Sutherland models [11]. Indeed, *for these models* thermal duality as discussed here follows from the known coupling-constant duality. The concordance of the general thermal duality for abstract g-on statistics with the more specific and complete duality for Calogero-Sutherland quanta vividly confirms the idea that these models embody ideal g-on statistics, subject to the subtlety noted above.

Remarks on the Mott problem.—As we have emphasized, the application of g-on statistics to one-dimensional systems, even for the soluble models where it is formally correct, is not straightforward. Indeed in one space dimension the application of Fermi statistics to derive the low-energy properties of systems of fermionic quasiparticles has to be carefully considered, since interactions can change the properties qualitatively. The Fermi liquid must be considered as one special case of the generic Luttinger liquid [12]. Thus, for example, the thermodynamic properties of edge excitations in the fractional quantized Hall states are not correctly reproduced by the ideal g-on formulas, even though the electrons are (for example) formally m-ons in the $\nu = 1/m$ state. Of course the bulk filling fraction is nicely consistent with $1/g$ filling of the magnetic band, but this is a much weaker statement.

More speculative, but if correct probably more useful, is the possibility of applications to systems in higher dimensions. For in higher dimensions the phase space arguments of Landau apply, as in his justification of Fermi liquid theory, and make it plausible that (unlike in one space dimension) the approximation of noninteracting quasiparticles is accurate at low temperature.

There is a class of insulating materials, the Mott insulators, which are anomalous from the point of view of band or Fermi liquid theory. They are insulators when their valence band is precisely half filled. From the point of view of this paper, it is natural to hypothesize that in these materials the electrons are behaving as 2-ons [13]. The most important qualitative feature of Mott insulators, that is the existence of a gap at exactly half filling, follows directly. Such behavior is suggested, but certainly *not* proved, by the idea (formalized in the t-J model) that because of strong on-site repulsion a single electron excludes two states—namely states of both spins—from its lattice site. As we have taken pains to emphasize what is needed is local repulsion in momentum space, which could arise directly from a long-range force or indirectly through correlation effects.

In any case, our hypothesis leads to the statistical-mechanical consequences derived earlier, which could be tested in experiments or by numerical work on models. Most interesting are effects which arise just *below* half filling. The central consequence is the existence at $T = 0$ of a Fermi surface of anomalous size, and with anomalous values of the specific heat, Pauli susceptibility, etc. There *are* quantitative anomalies in the experimentally observed normal state specific heat of the CuO-based superconductors [14], and in the size of the "Fermi surface" in the t-J model as calculated using high-temperature expansions [15] and variationally [16]. (In these calculations, the nominal Fermi surface is identified from a strong feature in the density-density correlation function.) These anomalies are at least roughly consistent with the 2-on hypothesis: the specific heat is substantially larger, and the volume of the Fermi surface is roughly twice as large, as would be expected for ordinary fermions.

The research of C. N. was supported in part by a Fannie and John Hertz Foundation fellowship. The research of F. W. was supported in part by DOE Grant No. DE-FG02-90ER40542.

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