Self-consistent Analytical Model of the Rayleigh-Taylor Instability in Inertial Confinement Fusion

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The model presented overcomes past inconsistencies by applying asymptotic techniques. The obtained growth rate, $\gamma \approx \alpha(k)\sqrt{kg} - 2kv_a$ (where $v_a =$ ablation velocity, $\alpha(k) \equiv \sqrt{1 - (k/k_c)^r}$ represents the stabilization heat conduction effect, and the cutoff wave number k_c is much smaller than the inverse of the density scale length at the ablation front), reproduces numerical simulations and experiments in a more complete way than the so-called Takabe formula, $\gamma = 0.9\sqrt{kg} - 3kv_a$.

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The Rayleigh-Taylor instability (RTI) in inertial confinement fusion (ICF) [1] is critical for the achievement of appropriate implosions. In 1974, Bodner [2] reported a simple discontinuity model of such instability, however, he needed to introduce an *ad hoc* assumption to close the problem. He found that growth rate γ could be reduced below the classical value ($\sqrt{kg}$), due to mass ablation and described as $\gamma \approx \sqrt{kg} - kv_a$, where k is the transverse wave number, v_a is the flow velocity across the ablation front, and g is the target acceleration. However, the numerical simulations [3] and experimental results [4] which followed suggested larger stabilization effects.

Numerous attempts have been made to develop an analytical model by means of a surface discontinuity (ablation surface) separating two uniform fluids [5]. Since this approach leaves the solution undetermined, other approximations circumvent these difficulties either by including in the analysis a layer with a diffuse boundary centered at the ablation front (of thickness the density gradient scale length) and solving numerically a linearized eigenvalue problem [6,7] or by using a WKB model [8] (assuming very small wavelength perturbations).

Although the general belief is that the reduction is due to mass ablation, the single *theoretical support* for such a large stabilization is the so-called Takabe formula [6]:

$$\gamma = 0.9\sqrt{kg} - 3kv_a\,, \tag{1}$$

which was obtained by means of a *numerical fitting,* and it is repeatedly referenced in the literature. Laser fusion simulations seem to agree well with this formula, however, a recent set of indirect-drive experiments conducted on the Nova laser [9] do not suggest a large stabilization ablative effect [factor 3 in Eq. (1)].

It is well known that flow expanding through an interface may require additional information besides the conservation relations across it [10]. Obtaining these additional conditions has been the most speculative part of the models about RTI in ICF and a recurrent inconsistency for the last twenty years, all this becomes worse by the extra difficulty of approximating the near plasma corona region, dominated by the inhomogeneity of the fluid, in a realistic way. The model presented here overcomes this obstacle by considering, on the one hand, the inner structure of a thin transition layer (ablation surface), as in studying the stability of slow-combustion fronts [11] and, on the other hand, by performing, in a rigorous way, an asymptotic matching (no jump conditions) to both sides of it [12]. The soundness of the physical model is based on both the assumption of a sharp ablation front and on the smallness of the Mach number of the flow through it.

For simplicity, I am considering a planar foil of thickness d (small compared with the radius of the shell if spherical), which is moving with acceleration g, due to the ablation pressure p_a, generated by the heat flux coming from the corona. This slab is continuously ablating with a mass ablation rate $\dot{m}$.

We use the same one-fluid equations as Bodner [2] (in the frame moving with the unperturbed ablation front), but we do not assume incompressibility and, moreover, the heat conduction is explicitly taken into account

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{2}$$

$$\partial(\rho\boldsymbol{v})/\partial t + \nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v}) = -\nabla(\rho T) + \rho\mathbf{g}\,, \tag{3}$$

$$\partial/\partial t(\rho\boldsymbol{v}^2/2 + 3\rho T/2) + \nabla\cdot [\rho\boldsymbol{v}(5T/2 + \boldsymbol{v}^2/2) - K\nabla T] = \rho\boldsymbol{v}\cdot\mathbf{g}\,. \tag{4}$$

We also assume the fluid to be a monatomic perfect gas; the temperature is conveniently normalized, $\mathbf{g} = g\mathbf{e}_x$, and K is the thermal conductivity. Eq. (4) does not contain an energy deposition term, for instance by means of laser energy, because the absorption region is located at a large distance from the ablation surface, compared with the wavelength perturbations I am considering.

One can distinguish three asymptotic regions to be matched: (i) a cold and adiabatically compressed zone (region 1) of thickness d, which presents a maximum of the density ρ_a, where its values for the pressure and temperature are p_a and T_a, respectively; (ii) an adjacent thin layer (region 2), where the material is being heated and expands towards the corona. The pressure is approximately constant, $\rho T \approx p_a$, but not the density, which is

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of the order of magnitude of ρ_a. The characteristic thickness of region 2 is $\Delta \equiv K_a/\dot{m}$ (~the gradient scale length at the ablation front) [7], with K_a the thermal conductivity at T_a and ρ_a. The characteristic Mach number of the flow velocity, $M_a \equiv v_a/\sqrt{p_a/\rho_a}$, is assumed to be a small parameter, with $v_a \equiv \dot{m}/\rho_a$ defining the ablation velocity. (iii) Finally, the hot plasma corona (region 3), where the flow reaches sonic conditions, and the order of magnitude of the characteristic density $\rho_c \equiv \rho_a M_a^2$, temperature velocity $T_c \equiv v_c^2 \equiv (p_a/\dot{m})^2$, and thickness L are obtained in a simple way through mass, momentum, and energy balance, $\dot{m}T_c \sim KT_c/L$ (the thermal conductivity is assumed $K = \overline{K}T^{n-m}/\rho^m$, $1 < m < 2$, and $5 < n < 8$ [13], in order to roughly describe other possible transport energy mechanisms, such as radiation; in laser fusion and for electronic heat conduction, $m \equiv 0$ and $n = \frac{5}{2}$):

$$L \equiv (p_a/\dot{m})^{2n} 2\overline{K}/(5n\dot{m}p_a^m), \quad (5)$$

thus, one obtains $L = 2\Delta/(5nM_a^{2n}) \gg \Delta$, and $\Delta \ll d \ll L$ typically.

Attention is restricted to instability wave numbers $k > 1/d$ such that the equilibrium does not change on the time scale of the growth rate. As it will be proved, stabilization is found (in the limit $M_a \to 0$) for wave numbers $kd \sim (kL)^{1/n}$. Typical targets could have ratio d/L values such that $kv_a^2/g \sim kdM_a^2 \ll 1$ (main assumption of the model), and therefore $k\Delta \ll 1$ (simulations of Gardner *et al.* [3] have $kdM_a^2 \sim 0.04$). Let p_{a1} be the perturbed ablation pressure due to the deformation x_a of the rippled ablation front; using mass, momentum, and energy conservation one can obtain the characteristic velocity v_* and temperature T_* at distance k^{-1} from the ablation surface inside region 3, $v_* \sim \dot{m}T_*/p_a \sim \dot{m}T_c/[p_a(kL)^{1/n}]$, so the perturbed ablation pressure gives us $p_{a1} \sim p_a k x_a/(kL)^{1/n}$. Then, the heat conduction causes the pressure to increase on the crest of the rippled ablation front and decreases on the valley, damping the growth. Now, using a mechanical balance and $p_a \sim \rho_a g$, one can obtain, apart from numerical factors, $\partial^2 x_a/\partial t^2 \sim kg[1 - kd/(kL)^{1/n}]x_a$. Certainly, the convection of the material through the interface also damps the growth and, in some cases, both effects could be numerically comparable.

I look for solutions of perturbed quantities of the form $\exp(\gamma t + iky)$. Let $x = 0$, the unperturbed position of the ablation front, and $x_a \approx \xi \exp(\gamma t + iky)$, the perturbation (in the limit $k\Delta \to 0$, region 2, compared with regions 1 and 3, becomes a surface discontinuity). It is convenient to strain the x coordinate [12], defining a new variable in the form $s = x - x_a$ and to expand velocity $\boldsymbol{v} = v_{1y}(s)\exp(\gamma t + iky)\mathbf{e}_y + \{v_0(s) + [v_{1x}(s) + \gamma\xi]\exp(\gamma t + iky)\}\mathbf{e}_x$, density $\rho = \rho_0(s) + \rho_1(s)\exp(\gamma t + iky)$, and similarly for the temperature T (notice that v_{1x} is the velocity with respect to the moving perturbed ablation surface which is located at $s = 0$). We would then proceed with the analysis of Eqs. (2)–(4) in zero order and linearize in the first-order perturbations.

In region 1 ($s < 0$), after a transient time, since the flow velocity is very subsonic ($M_a \ll 1$), we have $p_a \approx \int_{-d}^{0} \rho_0(s')gds' \equiv \rho_{\rm av}gd$, with $\rho_{\rm av}$ the slab average density. Then, expanding for $|s/d| \ll 1$, one obtains $\rho_0 T_0 \approx p_a[1 + \rho_a s/(\rho_{\rm av}d)]$, $\rho_0/\rho_a \approx 1 + O(s/d)$, and $v_0/v_a \approx 1 + O(s/d)$. As shown by Bodner [see Eqs. (10)–(12) of Ref. [2]], the perturbed quantities satisfy the relations $\rho_1/\rho_0 = 2T_1/(3T_0) \approx 0$, $v_{1y} \approx i(C_1/\rho_a + \gamma\xi)$, and

$$C_2 \approx \xi\rho_a g - \gamma\xi\dot{m}(1 + \gamma/kv_a) + C_1 v_a(1 - \gamma/kv_a), \quad (6)$$

with C_1 and C_2 being the perturbed mass flow rate and momentum flux in the x direction at $s = 0^-$, respectively (the differences with respect to Bodner [2] in the ξ terms are due to the different definition of v_{1x} and the strain of the x coordinate).

In region 2, the zero-order solution matching to region 1 yields $\rho_0 T_0 \approx p_a$, $\rho_0 v_0 \approx \dot{m}$ and

$$5\dot{m}(T_0 - T_a)/2 \approx \overline{K}(T_0^n/p_a^m)(dT_0/ds), \quad (7)$$

where terms of the order of M_a^2 and Δ/d (gravity effects) have been neglected (notice that $T_0 \sim s^{1/n}$ for s/Δ is large). Perturbed quantities can be obtained in a simple way through mass, momentum, and energy conservation. Since $k\Delta \ll 1(\gamma\Delta/v_a \ll 1)$, it is straightforward to obtain the solution matching (at $s/\Delta \to -\infty$ but $|ks| \ll 1$) to region 1:

$$\rho_1 v_0 + \rho_0 v_{1x} \approx C_1, \quad (8)$$

$$\rho_1 T_0 + \rho_0 T_1 + 2\rho_0 v_0 v_{1x} + \rho_1 v_0^2 \approx C_2, \quad (9)$$

$$iv_{1y} \approx k\xi(v_0 - v_a - \gamma/k) - C_1/\rho_a, \quad (10)$$

$$T_1 \approx (T_0 - T_a)[C_3 + 5C_1 s/(\dot{m}\Delta)](T_a/T_0)^n, \quad (11)$$

with C_3 being an arbitrary constant.

To analyze region 3 ($s > 0$), only a layer of thickness $s \sim k^{-1} \ll L$ needs to be considered; let then $\eta \equiv ks$ be the normalized space variable in this region. The asymptotic analysis is complex and requires a power expansion with respect to two assumed small parameters: $\delta \equiv (kL)^{-1/n}$ (characteristic Mach number squared in such a layer v_*^2/T_*) and $\varepsilon \equiv M_a(kL)^{1/2n}$ ($\sim \gamma/kv_* \sim \sqrt{kv_a^2/g}$). The dominant term of the unperturbed solution matching the solution of the region 2 at $s/\Delta \to +\infty$ is $\rho_c/\rho_0 = v_0/v_c = T_0/T_c \approx \delta\eta^{1/n} + \cdots$. We then expand the perturbations as follows: $\rho_1 T_0 + \rho_0 T_1 \approx \delta p_a(p_1 + \varepsilon p_2 + \delta p_3 + \cdots)$, $v_{1x} \approx \delta v_c (u_1 + \varepsilon u_2 + \delta u_3 + \cdots)$, $v_{1y} \approx \delta v_c(v_1 + \varepsilon v_2 + \delta v_3 + \cdots)$, and $T_1 \approx \delta T_c(\theta_1 + \varepsilon\theta_2 + \delta\theta_3 + \cdots)$, where $w_j \equiv \{p_j, u_j, v_j, \theta_j, d\theta_j/d\eta\}$ are functions of η. Then, Eqs. (2)–(4) lead to a fifth order system of linear differential equations with variable coefficients for the perturbations in each order $[O(1),\ O(\delta)$, and $O(\varepsilon)]$ of the type $dw_j/d\eta = a_0 w_j + c_j w_1 + b_j k\xi$, a_0 and b_j, c_j are matrix and vector coefficients (with $c_1 \equiv 0$), respectively, which depend on η.

In order to connect regions 2–3 we match at the different orders the solutions of Eqs. (8)–(11), taking the limit $s/\Delta \rightarrow +\infty$ (with $ks \ll 1$), to p_j, θ_j, u_j, v_j at $\eta \rightarrow 0^+$. Then Eqs. (10) and (11) lead to

$$iv_{1y}/v_c|_{s/\Delta\rightarrow+\infty} \approx \delta k\xi\eta^{1/n} - \gamma\xi M_a^2/kv_a + \cdots, \quad (12)$$

$$T_1/T_c|_{s/\Delta\rightarrow+\infty} \approx \delta C_1\eta^{1/n}/n\dot{m} + \cdots. \quad (13)$$

Equations (12) and (13), together with momentum and mass conservation, Eqs. (8) and (9), is all we need to connect regions 2–3. The solution $w \equiv w_1 + \varepsilon w_2 + \delta w_3 + \cdots$ will linearly depend on the constants C_1, C_2 and ξ. Moreover, it has two unbounded modes as $\eta \rightarrow +\infty$ [7], so the condition that solution must be bounded will determine C_1 and C_2. Both modes explode as $\exp(\eta)$ [for instance $\theta_j \sim \eta^{(2\pm\sqrt{5})/(4n)}\exp(\eta)[1 + O(1/\eta) + \cdots]$] [14], this making the numerical determination of C_1 and C_2 troublesome (this last point has been always unsatisfactorily treated in the past [6,7]). Now expanding $C_2/\delta p_a k\xi \equiv q$ and $C_1\dot{m}k\xi \equiv f$ one obtains

$$q \approx q_1 + q_2\frac{\gamma}{kv_a}M_a^2(kL)^{1/n} + q_3(kL)^{-1/n} + \cdots, \quad (14)$$

and similarly for f. The constants f_j, q_j are determined at each order imposing the vanishing of perturbed solution as $\eta \rightarrow +\infty$. Some numerical values are shown in Table I. Then, using Eq. (6) we obtain the dispersion relation

$$\gamma^2 + kv_a(1 + f)\gamma - (kv_a)^2 f - kg\left(1 - \frac{(\rho_{\rm av}/\rho_a)kd}{(kL)^{1/n}}\,q\right) = 0. \quad (15)$$

I claim that, since f and q are positive, the mass flow across the ablation front alone could never completely suppress the instability, but it is the last term in Eq. (15) that produces it. Then, retaining up to the first order in δ and ε, the unstable root is

$$\gamma \approx \sqrt{kg\left[1 - \frac{(\rho_{\rm av}/\rho_a)kd}{(kL)^{1/n}}\left(q_1 + \frac{q_3}{(kL)^{1/n}}\right)\right]} - kv_a(1 + f_1 + q_2)/2, \quad (16)$$

We point out that the factor multiplying kv_a above, $(1 + f_1 + q_2)/2$, takes a value of approximately 2 for all the range of interest in n and m values (becoming a general feature of the ablatively accelerated targets), with q_2, the *pressure contribution,* being the largest. Strictly, in the asymptotic limit $M_a \rightarrow 0$, the growth is stabilized because γ, the square root term in Eq. (16), vanishes for a cutoff wave number $k_c \approx [\rho_a L^{1/n}/(q_1\rho_{\rm av}d)]^{n/(n-1)}$, with k_c^{-1} much larger than the thickness Δ of the ablation front. For larger wave numbers the modes become oscillatory, similar to the effect produced by surface tension in liquids. The stabilizing effect of second and first terms of Eq. (16) would increase substantially by means of reduced values of the peak density ρ_a (for $\dot{m}$ fixed) and L, respectively. Eq. (16) has been rigorously derived without any assumption regarding the behavior of the far away plasma corona or laser energy deposition region, which affects the determination of L. On the other hand, L is obtained using p_a and $\dot{m}$ values which can be derived from either one-dimensional numerical simulations, experiments or analytical models. For instance, some known scaling laws for planar analytical models in laser fusion ($p_a \sim I_L^{2/3}\lambda_L^{-2/3}, \dot{m} \sim I_L^{1/3}\lambda_L^{-2/3}$) [15] lead to $L \sim I_L^{4/3}\lambda_L^{14/3}$, being I_L = laser intensity and λ_L = laser wavelength. On the other hand, spherical analytical models [6,7], if energy is absorbed in the supersonic region of the corona, lead to L/d, depending basically on the ratio of the sonic to the ablation density and the aspect ratio.

In Fig. 1 the numerical results calculated by Takàbe [6] in the laser fusion case (solid squares), using the same normalization, for several ratios of the ablation to the sonic densities ρ_a/ρ_s, are compared with the results from Eq. (16) ($k = l/r_a$, where l is the Legendre index of spherical harmonics, r_a is the ablation radius, $G = gr_s/C_s^2$, C_s is the isothermal sound velocity at the sonic radius r_s, and $\Upsilon \equiv \gamma C_s/r_s$).

Figure 2 next shows the comparison with some results of the numerical simulations reported by Gardner *et al.* [3]. L has been obtained from some of the data reported there; plastic CH targets of thickness 100 μm exposed to blue (0.26 μm) or red (1.06 μm) laser radiation (3×10^{14} Wcm^{-2}). One of them, the corresponding to critical dump (all the laser energy was deposited at critical density

TABLE I. Coefficients f_j and q_j for several values of n and m.

n/m	f_1	f_2	f_3	q_1	q_2	q_3
2.5/0	1.03	−0.52	−0.01	0.67	2.08	0.61
5/1	1.01	−0.26	0.23	0.80	2.03	1.29
7/1.5	1.00	−0.19	0.44	0.85	2.02	1.76

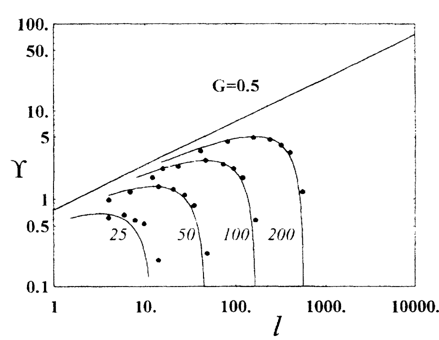

FIG. 1. Normalized growth rate versus Legendre index for several values of ρ_a/ρ_s. Curves were obtained from Eq. (16) in the text and solid squares are points calculated by Takabe [6]. The straight line corresponds to the classical growth $\sqrt{kg}$ in normalized unit.

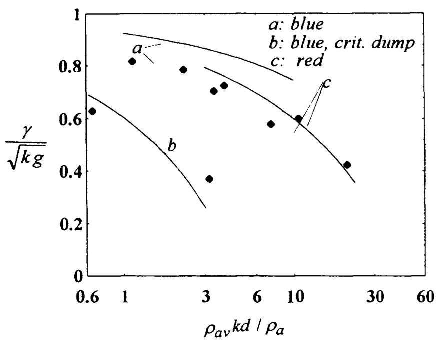

FIG. 2. Normalized growth rate versus normalized wave number. Curves were obtained from Eq. (16) and solid squares from the numerical simulations (points are joined to clarify the behavior).

in the simulation) which has a value for L much smaller than the others, shows a departure from the rest, in agreement with what is predicted by Eq. (16). Such a behavior cannot be only attributed to the mass ablation term of Eq. (1).

Finally, for the actual conditions of a set of indirect-drive experiments [$\Delta \sim 1 - 4\ \mu$m $\approx$ minimum density gradient scale length$\equiv LM_a^{2n}(n+1)^{n+1}/n^n$, $kv_a^2/g < 0.02$] [9], Eq. (16) leads to $\gamma \approx \sqrt{kg} - 2kv_a$, which shows a good agreement with the data of the experiment and the corresponding LASNEX simulation.

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