Distribution of "Level Velocities" in Quasi-1D Disordered or Chaotic Systems with Localization

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The explicit analytic expression for the distribution function of parametric derivatives of energy levels ("level velocities") is derived for the chaotic quantum systems belonging to the quasi-1D universality class (quantum kicked rotator, "domino" billiard, disordered wire, etc.).

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It is generally accepted nowadays that the problem of a quantum particle moving in a random potential first addressed in the context of Anderson localization has much in common with such problems in the domain of quantum chaos as "quantum kicked rotator" [1], the ionization of Rydberg atoms by a microwave radiation [2], and chains of quantum billiards (the "domino billiard" [3]). This analogy first suggested in [4] proved to be very fruitful for understanding the phenomenon of the so-called "dynamical localization" [1].

More recently it was found that there exists a convenient mathematical framework—the ensemble of random banded matrices (RBM), see [5,7] and references therein—generalizing the classical Gaussian ensembles of random matrices and allowing for a uniform description of the typical features common the above-mentioned systems. Investigation of the RBM ensemble helped to reveal a number of universal scaling relations characterizing statistics of energy levels and eigenfunctions of all these systems [1,5,6]. Another important feature is that the stochastic RBM model can be mapped onto a regular field-theoretical model—a so-called nonlinear graded σ model—allowing in some cases for an exact analytical treatment and so providing one with a powerful tool of research [6,7]. This nonlinear σ model turns out to be identical to that derived earlier by Efetov and Larkin in the course of study of the Anderson localization in disordered wires [8].

All these facts suggest the introduction of a notion of a "quasi-1D universality class" of disordered and chaotic systems. All statistical properties of systems belonging to this class are dependent on the only scaling parameter: the ratio $x = L/\xi$ between the sample length L and the localization length ξ. The explicit form of the scaling function was derived analytically for the so-called inverse participation ratio measuring the extent of eigenfunctions. For other quantities characterizing eigenfunction statistics, analytical results are available in two limiting cases $x \gg 1$ ($x \ll 1$) corresponding to the complete localization (delocalization) of eigenfunctions [6,7]. As to the statistics of energy levels, only heuristic expressions deduced from the numerical data were available so far [1].

Quite recently an interesting new development in the study of weakly disordered metallic systems and their chaotic counterparts has been made in a set of works by the MIT group [10]. Developing earlier ideas from the papers [11], the authors of [10] studied the energy level motion as a function of some external tunable parameter α. Physically the role of such a parameter can be played by, e.g., an external magnetic field, the strength of a scattering potential for disordered metal, a form of confining potential for quantum billiards, or any other appropriate parameter on which the system Hamiltonian is dependent. A high degree of universality in a "level response" of a generic chaotic system to an external perturbation has been revealed. It was found that a set of "level velocities" (LV) $v_n(\alpha) \equiv \partial E_n/\partial\alpha$, with index n labeling different energy levels, can be characterized after a proper normalization by universal correlation functions $\langle v_n(\alpha)v_{n'}(\alpha')\rangle$ whose form is dependent only on symmetries of the unperturbed Hamiltonian and those of perturbations. Related quantities characterizing energy level response, such as the distribution of "level curvatures" $K_n \equiv \partial^2 v_n/\partial\alpha^2$ and that of "avoided crossings" (local minima of adjacent level spacings) were studied in the papers [11,12]. Let us also mention an intimate connection between the level response characteristics and the system conductance if the role of the perturbation is played by the Aharonov-Bohm (AB) magnetic flux; see the detailed discussion of the issue in [13].

Most of the analytic work on the level response characteristics done so far has made use of the analogy between quantum chaotic systems and random matrix ensembles, which is now a commonly accepted principle in the domain of quantum chaos [14]. However, a simulation of chaotic (disordered) systems by the classical Gaussian random matrix ensembles (or equivalent models) precludes effects of Anderson localization from being taken into account. On the other hand, these effects should considerably modify all the results when one deals with systems belong to the quasi-1D universality class introduced above. This indeed was found to be the case in the numerical study of the curvature distribution for the *periodic* RBM simulating a disordered ring threaded by the AB flux [15].

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In this Letter I derive, for the first time, the distribution of the level velocities for the systems belonging to the quasi-1D universality class in the most interesting limit of infinite sample length when the role of the localization effects is expected to be maximal. As the particular model of unperturbed system I use the ensemble of *nonperiodic* Hermitian RBM simulating a quasi-1D system (an isolated piece of wire, an irregular billiard chain, etc.) subject to a sufficiently strong magnetic field. The class of perturbation considered corresponds to a slight change of scattering potential within the wire. For this reason obtained results could not be straightforwardly applied to a *periodic* geometry and AB flux playing a role of perturbation. The latter case calls for a separate consideration.

Let us consider an unperturbed chaotic or disordered isolated quasi-1D system of finite size L having $N \propto L$ energy levels E_n, $n = 1,2,\dots,N$ and described by a Hamiltonian $\mathcal{H}$ whose statistical properties are adequately simulated by those of the RBM ensemble. Let us study the level response to a perturbation $\delta\mathcal{H} = \alpha\mathcal{V}$, with α being a small parameter. For this purpose it is convenient to introduce the resolvent operator $R_\alpha^{\pm}(E) = [E - \mathcal{H} - \alpha\mathcal{V} \pm i\epsilon]^{-1}$, with ϵ being a positive infinitesimal. Then one has the following self-evident identity:

$$\lim_{\epsilon\to 0} \epsilon \, \mathrm{Tr} R_0^+(E) \, \mathrm{Tr} R_\alpha^-(E) = \lim_{\epsilon\to 0} \sum_{n,m} \frac{\epsilon}{[E - E_n(0) + i\epsilon][E - E_m(\alpha) - i\epsilon]} . \tag{1}$$

It is well known that for the chaotic quasi-1D system of any *finite* size the spectrum consists of the set of *nondegenerate* levels, the probability $\mathcal{P}(\delta E)$ for two adjacent levels to be separated by a gap δE tending to zero when $\delta E \to 0$ (see, e.g., the so-called "Izrailev distribution" [1] and the recent analytical results by Kolokolov on the related subject [9]). Let us now impose the following requirements: (i) the limiting procedure in Eq. (1) to be performed *prior* to the thermodynamic limit $L \to \infty$ and (ii) the parameter $\alpha \propto \epsilon \to 0$ when performing the limiting procedure. Then the only nonvanishing contribution to the double sum in Eq. (1) is given by pairs of coinciding level indices $n = m$. Introducing the notations $\mu = 2\epsilon/\alpha$ and $v_n = \partial E_n/\partial\alpha|_{\alpha=0}$ one obtains

$$K(\mu) = \lim_{\epsilon\to 0} \epsilon \mathrm{Tr} R_0^+(E) \mathrm{Tr} R_{2\epsilon/\mu}^-(E) = \pi \sum_n \frac{\mu^2 + i\mu v_n}{\mu^2 + v_n^2} \delta(E - E_n) . \tag{2}$$

Now performing formally the averaging over the ensemble of Hamiltonians $\mathcal{H}$ (denoted by the angular brackets $\langle\cdots\rangle$) and introducing the mean level density $\rho(E)$, one immediately finds the relation between $K(\mu)$ and the distribution $\mathcal{P}(v)$ of the level velocities:

$$\frac{1}{N\rho} \mathrm{Re} \langle K(\mu)\rangle = \pi\mu^2 \int_{-\infty}^{\infty} \frac{\mathcal{P}(v)}{v^2 + \mu^2} dv \equiv \pi\mu \int_0^{\infty} e^{-\mu k} dk \int_{-\infty}^{\infty} \mathcal{P}(v) \cos kv \, dv . \tag{3}$$

The relations presented above are actually valid for an arbitrary chaotic or disordered system. Let us now specify the Hamiltonian $\mathcal{H}$ as being a $N \times N$ Hermitian random matrix with independent Gaussian distributed entries $\mathcal{H}(i,j)$ with zero mean and the variance $\langle \mathcal{H}^*(i,j)\mathcal{H}(i,j)\rangle = a(|i - j|/b)[1 + \delta_{ij}]$. Here the function $a(r)$ is assumed to vanish at $r \to \infty$ at least exponentially fast, the parameter b (assumed to be large: $b \gg 1$) defining, therefore, the effective bandwidth of RBM.

In order to perform the ensemble average, I employ Efetov's supersymmetric approach [8]. The detailed exposition of the method as applied to the RBM ensemble can be found in [7] and is not repeated here. After performing all the necessary steps, the problem is mapped onto a 1D nonlinear σ model with the action $S[Q] = S_0[Q] + \delta S[Q]$, where

$$S_0[Q] = \frac{\gamma}{4} \sum_{i=1}^{N} \mathrm{Str}(Qi - Q_{i+1})^2 + i\pi\rho\epsilon \sum_{i=1}^{N} \mathrm{Str} Qi\Lambda ,$$
$$\delta S[Q] = \sum_{k=1}^{\infty} \frac{1}{k} \left(-\frac{2i\pi\rho\epsilon}{\mu}\right)^k \times \sum_{i_1,\dots,i_k} \mathcal{V}(i_1,i_2)\mathcal{V}(i_2,i_3)\cdots \mathcal{V} i_k, i_1) \mathrm{Str} \times \prod_{m=1}^{m=k} Q_{i_m} \frac{1-\Lambda}{2} . \tag{4}$$

Here Str stands for the supertrace [8], the 4×4 matrices Q_i belong to the graded coset space $U(1,1/2)/U(1,1) \times U(1,1)$ [7,8], and $\Lambda = \mathrm{diag}(1,1,-1,-1)$.

The nonlinear σ model with the action $S_0[Q]$ describes the statistical properties of the unperturbed quasi-1D system, and it was intensively studied in [6,7]. The main parameter is the coupling constant γ expressed in terms of the RBM parameters as follows: $\gamma = (\pi\rho)^2 \sum_r a(r) r^2 \propto b^2$ [6,7]. It defines the only characteristic spatial length scale due to disorder: the localization length $\xi \propto \gamma$. On the more formal level it plays the role of the correlation length of the matrix field Q_i. That means that the matrices Q_i and Q_j can be considered as equal to each other as long as $|i - j| \ll \gamma$. Let us now make a natural assumption that the spatial structure of the perturbation $\mathcal{V}$ is of the same type as that of the unperturbed Hamiltonian $\mathcal{H}$, i.e., $\mathcal{V}(i,j)$ vanishes sufficiently fast as long as $|i - j| \gg b$. We will call such a perturbation $\mathcal{V}$ the generic one. Then in view of the relation $1 \ll b \ll b^2 \propto \gamma$, one can put $Q_{i_1} = Q_{i_2} = \cdots = Q_{i_k}$ in the expression for $\delta S[Q]$.

From previous experience [6,7] one can anticipate that the main contribution to the integrals over the matrix field Q_i is coming from the asymptotic domain $Q_i \sim 1/\epsilon\gamma$ as long as $\epsilon \rightarrow 0$. As it will be clear afterwards, the essential values of the parameter μ are of the order of $\mu \sim \max\{\gamma^{-1/2}, N^{-1/2}\}$. For a generic random perturbation one notices that $\mathcal{V}_0 \equiv (\mathcal{V}^2)(i,i) = \sum_m |\mathcal{V}(i,m)|^2$ is a deterministic (self-averaging) quantity of the order of unity, whereas $\mathcal{V}(i,i)$ is a random quantity with zero mean and variance of the order of $\langle \mathcal{V}(i,i)^2\rangle \sim \mathcal{V}_0/b \ll \mathcal{V}_0$. Combining all these estimates together one arrives at the final form for the effective action of the nonlinear σ model:

$$S[Q] = S_0[Q] - \frac{1}{2}\left(\frac{\pi\rho\epsilon}{\mu}\right)^2 \mathcal{V}_0 \times \mathrm{Str}\sum_i Q_i(1-\Lambda)Q_i(1-\Lambda). \tag{5}$$

In view of the local-in-space structure of the last term in Eq. (5), the corresponding integral over the matrices Q_i can be performed by the same transfer-matrix method used earlier [6,7] with minor modifications. As the result one finds $\frac{1}{\rho N}\langle K(\mu)\rangle = \frac{1}{x} I(x)$ where

$$I(x) = \int_0^x d\tau_1 \int_0^{x-\tau_1} d\tau \int_0^\infty \frac{dy}{y} Y^{(1)} \times (x - \tau - \tau_1; y) Y^{(2)}(\tau, \tau_1; y), \tag{6}$$

with both functions $Y^{(1)}(\tau, y)$ and $Y^{(2)}(\tau, \tau_1, y)$ satisfying the same differential equation:

$$\frac{\partial Y}{\partial \tau} = \mathcal{G} Y; \qquad \mathcal{G} = y^2 \frac{\partial 2}{\partial y^2} - \left(y + \frac{y^2}{4g^2}\right). \tag{7}$$

Here the "scaling" notations $g = (\mu/\pi\rho)[\gamma/2\mathcal{V}_0]^{1/2}$ and $x = N/2\gamma$ were introduced for the sake of convenience. Equation (7) should be supplied with the "initial" conditions:

$$Y^{(1)}(\tau = 0; y) = 1, \qquad Y^{(2)}(\tau = 0, \tau_1; y) = y Y^{(1)}(\tau_1; y), \tag{8}$$

the first one corresponding to the elastic reflection of the quantum particle at the sample edges and the second one related to the details of the transfer matrix method [7].

In order to be able to deal with the function $I(x)$ efficiently, it is more convenient to consider its Laplace transform $I_L(p) = \int_0^\infty e^{-px} I(x) dx$. One finds

$$I_L(p) = \int_0^\infty \frac{dy}{y} Y_L^{(1)}(p; y) Y_L^{(2)}(p; y), \tag{9}$$

where the functions $Y_L^{(1),(2)}(p, y)$ satisfy the system of two ordinary differential equations:

$$\mathcal{G}_p Y_L^{(1)}(p; y) = -\frac{1}{y^2}, \quad \mathcal{G}_p Y_L^{(2)}(p; y) = -\frac{1}{y} Y_L^{(1)}(p; y);$$
$$\mathcal{G}_p = \frac{\partial^2}{\partial y^2} - \left(\frac{1}{4g^2} + \frac{1}{y} + \frac{p}{y^2}\right). \tag{10}$$

One can write down the explicit solution to these equations noticing that the two Whittaker functions $W_{-g,\frac{\kappa}{2}}(y/g) \equiv \phi_1^\kappa(y)$ and $M_{-g,\frac{\kappa}{2}}(y/g) \equiv \phi_2^\kappa(y)$ are eigenfunctions of the operator $\mathcal{G}_p$ corresponding to the eigenvalue $\lambda_\kappa = -\frac{1}{4}(4p + 1 - \kappa^2)$, and the corresponding Wronskian is equal to $w_\kappa = -\Gamma[\kappa + 1]/\Gamma[g + (\kappa + 1)/2]$, where $\Gamma[z]$ stands for the Euler gamma function. Thus, one finds $Y_L^{(2)}(p; y) = \mathcal{R}\{y Y_L^{(1)}(p; y)\}$, where

$$Y_L^{(1)}(p; y) = \frac{1}{p}\Gamma[1 + g] W_{-g,1/2}(y/g) + \mathcal{R}\{1 - \Gamma[1 + g] W_{-g,1/2}(y/g)\}, \tag{11}$$

and the action of the opertor $\mathcal{R}$ on any function $f(y)$ is defined by ($\nu = \sqrt{4p + 1}$):

$$\mathcal{R}\{f\} = \frac{-1}{w_\nu}\left\{\phi_1^\nu(y) \int_0^y \phi_2^\nu(z) f(z) \frac{dz}{z^2} + \phi_2^\nu(y) \int_y^\infty \phi_1^\nu(z) f(z) \frac{dz}{z^2}\right\}. \tag{12}$$

Expressions (10)–(12) provide a formal possibility to find the function $I_L(p)$ for an arbitrary value of p. Actually, however, manageable expressions could be extracted only in the limiting case $p \rightarrow 0$ physically corresponding to the system length L being much larger than the localization length ξ, i.e., $x \propto L/\xi \gg 1$. The main simplification occurs if one notices that one can neglect the second term in the expression for $Y^{(1)}(p; y)$, Eq. (11), provided $p \rightarrow 0$. As a result Eqs. (9) and (10) can be presented in the form

$$I_L(p \rightarrow 0) = -g \frac{\Gamma^2[1 + g]}{p^2} \int_0^\infty \frac{dz}{z} W_{-g,1/2}(z) \tilde{Y}^{(2)}(z),$$
$$\mathcal{L}\, \tilde{Y}^{(2)}(z) = \frac{1}{z} W_{-g,1/2}(z), \quad \mathcal{L} = \frac{\partial^2}{\partial z^2} - \left(\frac{1}{4} + \frac{g}{z}\right). \tag{13}$$

Differentiating the identity $\mathcal{L}\, W_{-g,1/2}(z) = 0$ over the parameter g and using the condition $\tilde{Y}^{(2)}(z \rightarrow 0) = 0$ one obtains

$$\tilde{Y}^{(2)}(z) = \left[\frac{\partial}{\partial g} + \psi(1 + g)\right] W_{-g,1/2}(z),$$

where

$$(z) = \frac{\partial \ln \Gamma[z]}{\partial z}. \tag{14}$$

Substituting this expression into Eq. (13) and remembering the relation between $\langle K(\mu)\rangle$ and $I(x)$ one finds

$$\frac{1}{N\rho}\langle K(\mu)\rangle|_{x\rightarrow\infty} = g\left[\frac{\partial\psi(g)}{\partial g} + \frac{1}{2} g \frac{\partial^2\psi(g)}{\partial g^2}\right] \equiv \frac{g}{2}\int_0^\infty dk\, e^{-gk}\left[\frac{k/2}{\sinh(k/2)}\right]^2. \tag{15}$$

Introducing now the scaled level velocity $v_s = v\,\frac{1}{\pi\rho}\,(\frac{\gamma}{2V_0})^{1/2}$ and comparing Eq. (15) with Eq. (3), one restores the LV distribution function $\mathcal{P}(v_s)$ from its Fourier transform:

$$\mathcal{P}(v_s) = \int_0^\infty \frac{dk}{2\pi}\cos kv_s\left[\frac{k/2}{\sinh(k/2)}\right]^2 = \frac{\pi}{\sinh^2(\pi v_s)}\{\pi v_s \coth(\pi v_s) - 1\}. \quad (16)$$

This expression gives the explicit form of the LV distribution for the case of long quasi-1D disordered or chaotic system and is the main result of the present Letter.

Let us briefly mention that in the opposite limiting case of short systems whose length $L \ll \xi$ the effects of localization play no role, and one easily reproduces the Gaussian LV distribution typical for the chaotic systems studied in earlier papers [10,12]. This fact can be most easily checked by noticing that the solution to Eq. (7) in the domain $0 \leq \tau \leq x \ll 1$ is given by the expression

$$Y(\tau; y) = Y(\tau = 0; y)\exp -\tau\left(y + \frac{y^2}{4g^2}\right) \quad (17)$$

that immediately produces the required result when substituted for Eqs. (3) and (6).

Comparing the two limiting cases one concludes that the level velocities fluctuate much stronger when eigenfunctions are localized: (i) the probability of finding values of LV exceeding the typical value $\langle v^2\rangle^{1/2}$ decays in the case of extended states like e^{-cv^2}, i.e., much faster than a simple exponential typical for localized states, see Eq. (16), and (ii) the mean square $\langle v^2\rangle$ is proportional to the inverse localization length $1/\xi \propto 1/\gamma$ when localization takes place, i.e., is much larger than the value of the order of inverse system size $1/L \propto 1/N$ typical for systems with extended states and the same number of levels N. To this end it is interesting to note that in the papers [10] the quantity $\langle v^2\rangle$ was called the "generalized conductance" in view of its meaning for the AB case [13]. The results obtained in the present paper suggest that the level velocity of a disordered system subject to a random perturbation is rather related to the so-called "inverse participation ratio" which is inversely proportional to the eigenfunction extent. A more detailed discussion of this issue will be published elsewhere [16].

It is interesting to check all these predictions by a direct numerical simulation of the systems belonging to the quasi-1D universality class. In particular, for a chain of chaotic quantum billiards (the domino billiard [3]) one should be able to observe a substantial increase in LV fluctuations when passing from the regular chain to an irregular one.

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