

# Exact Results for Hamiltonian Walks from the Solution of the Fully Packed Loop Model on the Honeycomb Lattice

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We derive the nested Bethe ansatz solution of the fully packed  $O(n)$  loop model on the honeycomb lattice. From this solution we derive the bulk free energy per site along with the central charge and geometric scaling dimensions describing the critical behavior. In the  $n = 0$  limit we obtain the exact compact exponents  $\gamma = 1$  and  $\nu = 1/2$  for Hamiltonian walks, along with the exact value  $\kappa^2 = 3\sqrt{3}/4$  for the connective constant (entropy). Although having sets of scaling dimensions in common, our results indicate that Hamiltonian walks on the honeycomb and Manhattan lattices lie in different universality classes.

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The configurational statistics of polymer chains have long been modeled by self-avoiding walks. In the low-temperature limit, the enumeration of a single self-attracting polymer in dilute solution reduces to that of compact self-avoiding walks. A closely related problem is that of Hamiltonian walks in which the self-avoiding walk visits each site of a given lattice and thus completely fills the available space. Hamiltonian walks are directly related to the Gibbs-DiMarzio theory for the glass transition of polymer melts [1]. More than thirty years ago now, Kasteleyn obtained the exact number of Hamiltonian walks on the Manhattan oriented square lattice [2]. More recently this work has been significantly extended to yield exactly solved models of polymer melts [3]. The critical behavior of Hamiltonian walks on the Manhattan lattice has also been obtained from the  $Q = 0$  limit of the  $Q$ -state Potts model [4]. In particular, this Hamiltonian walk problem has been shown [3,4] to lie in the same universality class as dense self-avoiding walks, which follow from the  $n = 0$  limit in the low-temperature or densely packed phase of the honeycomb  $O(n)$  model [5, 6].

As exact results for Hamiltonian walks are confined to the Manhattan lattice, the behavior of Hamiltonian walks on nonoriented lattices, and the precise scaling of compact two-dimensional polymers, remains unclear [7–9]. The exact value of the (Hamiltonian) geometric exponent  $\gamma^H$  was conjectured to be  $\gamma^H = \gamma^D$ , where  $\gamma^D = \frac{19}{16}$  was extracted via the Coulomb gas method for dense self-avoiding walks [6,8]. However, recent numerical investigations of the collapsed and compact problems are more suggestive of the value  $\gamma^H = 1$  [9–11].

More recently, Blöte and Nienhuis [12] have argued that a universality class different from dense walks governs the  $O(n)$  model in the zero temperature limit (the fully packed loop model). Based on numerical evidence obtained via finite-size scaling and transfer matrix techniques, along with a graphical mapping at  $n = 1$ , they argued that the model lies in a new universality class char-

acterized by the superposition of a low-temperature  $O(n)$  phase and a solid-on-solid model at a temperature independent of  $n$ . This model is identical to the Hamiltonian walk problem in the limit  $n = 0$ . In this Letter we present exact results for Hamiltonian walks on the honeycomb lattice from an exact solution of this fully packed loop model. We derive the physical quantities which characterize Hamiltonian walks on the honeycomb lattice. These include a closed form expression for the connectivity, or entropy, and an exact infinite set of geometric scaling dimensions which include a conjectured value by Blöte and Nienhuis [12]. Our results settle the above-mentioned controversy in favor of the universal value  $\gamma^H = 1$ .

In general, the partition function of the  $O(n)$  loop model can be written as

$$Z_{O(n)} = \sum t^{\mathcal{N} - \mathcal{N}_b} n^{\mathcal{N}_L}, \quad (1)$$

where the sum is over all configurations of closed and nonintersecting loops covering  $\mathcal{N}_b$  bonds of the honeycomb lattice, and  $\mathcal{N}$  is the total number of lattice sites (vertices). Here the variable  $t$  plays the role of the  $O(n)$  temperature,  $n$  is the fugacity of a closed loop, and  $\mathcal{N}_L$  is the total number of loops in a given configuration.

For the particular choice  $t = t_c$ , where [5]

$$t_c^2 = 2 \pm \sqrt{2 - n}, \quad (2)$$

the related vertex model is exactly solvable with a Bethe ansatz type solution for both periodic [13–15] and open [16] boundary conditions. This critical line is depicted as a function of  $n$  in Fig. 1. Here we extend the exact solution curve along the line  $t = 0$ , where the only nonzero contributions in the partition sum (1) are for configurations in which each lattice site is visited by a loop, i.e., with  $\mathcal{N} = \mathcal{N}_b$  [17]. This is the fully packed model recently investigated by Blöte and Nienhuis [12].

We consider a lattice of  $\mathcal{N} = 2MN$  sites as depicted in Fig. 2, i.e., with periodic boundaries across a finite strip of width  $N$ . The allowed arrow configurations and the

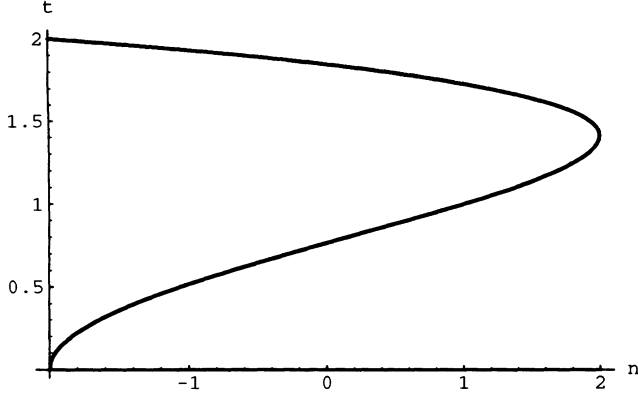


FIG. 1. The exact solution curve  $t = t_c(2)$  of the  $O(n)$  model on the honeycomb lattice as a function of  $n$ . In this Letter we extend the exact solution curve along the line  $t = 0$ .

$$e^{i\epsilon} \left[ -\frac{\sinh\left(\theta_\alpha - i\frac{\lambda}{2}\right)}{\sinh\left(\theta_\alpha + i\frac{\lambda}{2}\right)} \right]^N = -\prod_{\mu=1}^{r_2} -\frac{\sinh\left(\theta_\alpha - \phi_\mu + i\frac{\lambda}{2}\right)}{\sinh\left(\theta_\alpha - \phi_\mu - i\frac{\lambda}{2}\right)} \prod_{\beta=1}^{r_1} \frac{\sinh(\theta_\alpha - \theta_\beta - i\lambda)}{\sinh(\theta_\alpha - \theta_\beta + i\lambda)}, \quad \alpha = 1, \dots, r_1. \quad (4)$$

$$e^{i\epsilon} \prod_{\alpha=1}^{r_1} -\frac{\sinh\left(\phi_\mu - \theta_\alpha - i\frac{\lambda}{2}\right)}{\sinh\left(\phi_\mu - \theta_\alpha + i\frac{\lambda}{2}\right)} = -\prod_{\nu=1}^{r_2} \frac{\sinh(\phi_\mu - \phi_\nu - i\lambda)}{\sinh(\phi_\mu - \phi_\nu + i\lambda)}, \quad \mu = 1, \dots, r_2. \quad (5)$$

Here the seam parameter  $\epsilon = \lambda$  for the largest sector, and  $\epsilon = 0$  otherwise. Apart from the seam, this exact solution on the honeycomb lattice follows from earlier work by Baxter on the colorings of the hexagonal lattice [19]. Baxter derived the Bethe ansatz solution and evaluated the bulk partition function per site in the region  $n \geq 2$ . The corresponding vertex model was later considered in the region  $n < 2$  with regard to the polymer melting transition at  $n = 0$  [20].

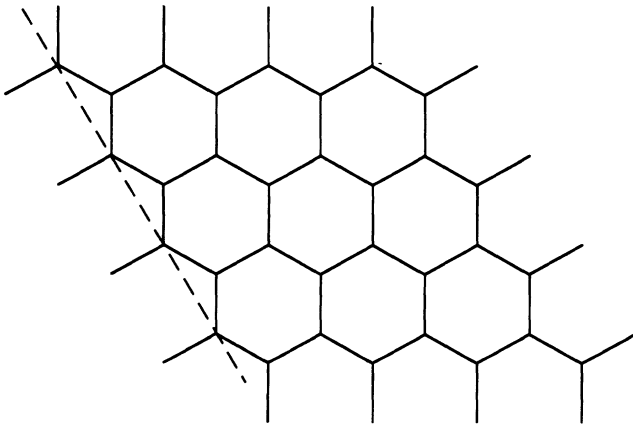


FIG. 2. The periodic honeycomb lattice of width  $L$ . Dashed lines indicate the position of the seam.

corresponding weights of the related vertex model are shown in Fig. 3. Here the parameter  $n = s + s^{-1} = 2 \cos \lambda$ . In Fig. 2 we also show a seam to ensure that loops which wrap around the strip pick up the correct weight  $n$  in the partition function. The corresponding weights along the seam are also given in Fig. 3 [18]. We find that the eigenvalues of the row-to-row transfer matrix of the vertex model are given by

$$\Lambda = \prod_{\alpha=1}^{r_1} -\frac{\sinh\left(\theta_\alpha - i\frac{\lambda}{2}\right)}{\sinh\left(\theta_\alpha + i\frac{\lambda}{2}\right)} \prod_{\mu=1}^{r_2} -\frac{\sinh(\phi_\mu + i\lambda)}{\sinh \phi_\mu} + e^{i\epsilon} \prod_{\mu=1}^{r_2} -\frac{\sinh(\phi_\mu - i\lambda)}{\sinh \phi_\mu}, \quad (3)$$

where the roots  $\theta_\alpha$  and  $\phi_\mu$  follow from

More generally, the fully packed loop model can be seen to follow from the honeycomb limit of the solvable square lattice  $A_2^{(1)}$  loop model [21,22]. Equivalently, the related vertex model on the honeycomb lattice is obtained in the appropriate limit of the  $A_2^{(1)}$  vertex model on the square lattice in the ferromagnetic regime. This latter model is the  $su(3)$  vertex model [23]. One can verify that the above results follow from the honeycomb limit of the algebraic Bethe ansatz solution of the  $su(3)$  model [24] with appropriate seam. It should be noted that

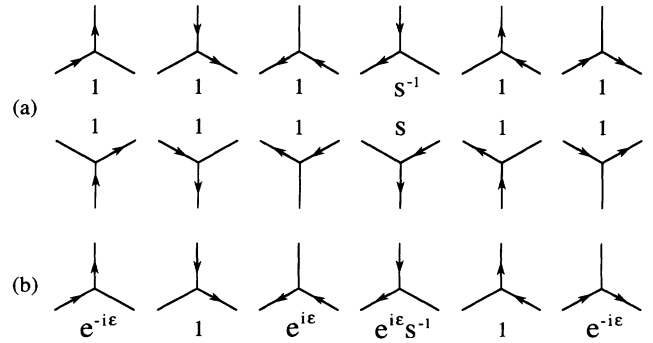


FIG. 3. (a) The allowed arrow configurations and their corresponding vertex weights. (b) The modified vertex weights along the seam.

Reshetikhin [22] has performed similar calculations to those presented here, although in the absence of the seam, which plays a crucial role in the underlying critical behavior.

Defining the finite-size free energy as  $f_N = N^{-1} \ln \Lambda_0$ , we derive the bulk value to be

$$f_\infty = \int_{-\infty}^{\infty} \frac{\sinh^2 \lambda x \sinh(\pi - \lambda)x}{x \sinh \pi x \sinh 3\lambda x} dx. \quad (6)$$

This result is valid in the region  $0 < \lambda \leq \pi/2$ , where the Bethe ansatz roots defining the largest eigenvalue  $\Lambda_0$  are all real. We note that the most natural system size  $N$  is a multiple of 3, for which the largest eigenvalue occurs with  $r_1 = 2N/3$  and  $r_2 = N/3$  roots. In the limit  $\lambda \rightarrow 0$ ,  $f_\infty$  reduces to the known  $n = 2$  value [19,13]

$$f_\infty = \ln \left[ \frac{3\Gamma^2(1/3)}{4\pi^2} \right]. \quad (7)$$

There is, however, a cusp in the free energy at  $\lambda = \pi/2$ . For  $\lambda > \pi/2$ , the largest eigenvalue has roots  $\theta_\alpha$  shifted by  $i\pi/2$ . The result for  $f_\infty$  is that obtained from (6) under the interchange  $\lambda \leftrightarrow \pi - \lambda$ , reflecting a symmetry between the regions  $-2 \leq n \leq 0$  and  $0 \leq n \leq 2$ . Thus the value (7) holds also at  $n = -2$ , in agreement with the  $t_c \rightarrow 0$  limiting value [13].

As our interest here lies primarily in the point  $n = 0$  ( $\lambda = \pi/2$ ), we confine our attention to the region  $0 \leq n \leq 2$ . At  $n = 0$ , we find that the above result for  $f_\infty$  can be evaluated exactly to give the partition sum per site,  $\kappa$ , as

$$\kappa^2 = 3\sqrt{3}/4, \quad (8)$$

and thus  $\kappa = 1.13975\dots$  follows as the exact value for the entropy or connective constant of Hamiltonian walks on the honeycomb lattice. This numerical value has been obtained previously via the same route in terms of an infinite sum [25]. Our exact result (8) is to be compared with the open self-avoiding walk, for which  $\mu^2 = 2 + \sqrt{2}$  [5], and so  $\mu = 1.84775\dots$ . It follows that for self-avoiding walks on the honeycomb lattice, the entropy loss per step due to compactness, relative to the freedom of open configurations, is exactly given by

$$\frac{1}{2} \ln \left[ \frac{3\sqrt{3}}{4(2 + \sqrt{2})} \right] = -0.483161\dots \quad (9)$$

The central charge  $c$  and scaling dimensions  $X_i$  defining the critical behavior of the model follow from the dominant finite-size corrections to the transfer matrix eigenvalues [26]. For the central charge,

$$f_N \approx f_\infty + \frac{\pi \zeta c}{6N^2}. \quad (10)$$

The scaling dimensions are related to the inverse correlation lengths via

$$\xi_i^{-1} = \ln(\Lambda_0/\Lambda_i) \approx 2\pi \zeta X_i/N. \quad (11)$$

Here  $\zeta = \sqrt{3}/2$  is a lattice-dependent scale factor.

The derivation of the dominant finite-size corrections via the Bethe ansatz solution of the vertex model follows

that given for the  $\text{su}(3)$  model in the antiferromagnetic regime [27] (see, also [28]). The derivation is straightforward though tedious, and we omit the details. In the absence of the seam, we find that the central charge is  $c = 2$  with scaling dimensions  $X = \Delta^{(+)} + \Delta^{(-)}$ , where

$$\Delta^{(\pm)} = \frac{1}{8} g \mathbf{n}^T C \mathbf{n} + \frac{1}{8g} (\mathbf{h}^\pm)^T C^{-1} \mathbf{h}^\pm - \frac{1}{4} \mathbf{n} \cdot \mathbf{h}^\pm, \quad (12)$$

$C$  is the  $\text{su}(3)$  Cartan matrix, and  $\mathbf{n} = (n_1, n_2)$  with  $n_1$  and  $n_2$  related to the number of Bethe ansatz roots via  $r_1 = 2N/3 - n_1$  and  $r_2 = N/3 - n_2$  [29]. The remaining parameters  $\mathbf{h}^\pm = (h_1^\pm, h_2^\pm)$  define the number of holes in the root distribution in the usual way [27]. We have further defined the variable  $g = 1 - \lambda/\pi$ .

With the introduction of the seam  $\epsilon = \lambda$ , we find that the central charge of the fully packed  $O(n)$  model is exactly given by

$$c = 2 - 6(1 - g)^2/g. \quad (13)$$

This is the identification made by Blöte and Nienhuis [12]. At  $n = 0$  we have  $g = 1/2$ , and thus  $c = -1$ . On the other hand, both Hamiltonian walks on the Manhattan lattice [3,4] and dense self-avoiding walks [6] lie in a different universality class with  $c = -2$ . However, as we shall see below, they do share common sets of scaling dimensions and thus critical exponents. This sharing of exponents between the fully packed and densely packed loop models has already been anticipated by Blöte and Nienhuis in their identification of the leading thermal and magnetic exponents [12]. Here we derive an exact infinite set of scaling dimensions.

Of most interest is the so-called watermelon correlator, which measures the geometric correlation between  $L$  nonintersecting self-avoiding walks tied together at their extremities  $\mathbf{x}$  and  $\mathbf{y}$ . It has a critical algebraic decay

$$\langle \phi_L(\mathbf{x}) \phi_L(\mathbf{y}) \rangle_c \sim |\mathbf{x} - \mathbf{y}|^{-2X_L}, \quad (14)$$

where  $X_L$  is the scaling dimension of the conformal source operator  $\phi_L(\mathbf{x})$  [6]. As along the line  $t = t_c$ , these scaling dimensions are associated with the largest eigenvalue in each sector of the transfer matrix. The pertinent scaling dimensions follow from the more general result

$$X = \frac{1}{2} g (n_1^2 + n_2^2 - n_1 n_2) - \frac{(1 - g)^2}{2g}. \quad (15)$$

The sectors of the transfer matrix are labeled by the Bethe ansatz roots via  $L = n_1 + n_2$ . The minimum scaling dimension in a given sector is given by  $n_1 = n_2 = k$  for  $L = 2k$  and  $n_1 = k - 1$ ,  $n_2 = k$  or  $n_1 = k$ ,  $n_2 = k - 1$  for  $L = 2k - 1$ . Thus we have the set of geometric scaling dimensions  $X_L$  corresponding to the operators  $\phi_L$  for the loop model,

$$X_{2k-1} = \frac{1}{2} g (k^2 - k + 1) - \frac{(1 - g)^2}{2g}, \quad (16)$$

$$X_{2k} = \frac{1}{2} g k^2 - \frac{(1 - g)^2}{2g}, \quad (17)$$

where  $k = 1, 2, \dots$ . The magnetic scaling dimension is given by  $X_\sigma = X_1$ , which agrees with the identification made in [12]. The eigenvalue related to  $X_2$  appears in the  $n_d = 2$  sector of the loop model, i.e., with two dangling bonds [12]. At  $n = 0$  this more general set of dimensions reduces to

$$X_{2k-1} = \frac{1}{4}(k^2 - k), \quad (18)$$

$$X_{2k} = \frac{1}{4}(k^2 - 1). \quad (19)$$

In comparison, the scaling dimensions for dense self-avoiding walks are [6]

$$X_L^{\text{DSAW}} = \frac{1}{16}(L^2 - 4). \quad (20)$$

Thus we have the relations

$$X_{2k-1} = X_{2k-1}^{\text{DSAW}} + \frac{3}{16}, \quad (21)$$

$$X_{2k} = X_{2k}^{\text{DSAW}}. \quad (22)$$

Note that  $X_1 = X_2 = 0$  and  $X_L > 0$  for  $L > 2$ . Identifying  $X_\epsilon = X_2$  as in [6], then the exponents  $\gamma = 1$  and  $\nu = 1/2$  follow in the usual way [30]. These are indeed the exponents to be expected for compact or space filling two-dimensional polymers.

The corresponding scaling dimensions for Hamiltonian walks on the Manhattan lattice are as given in (19) [4]. Exact Bethe ansatz results on this model indicate that the scaling dimensions  $X_\sigma = X_\epsilon = 0$ , from which one can also deduce that  $\gamma = 1$  and  $\nu = 1/2$  [31]. We also expect these results to hold for Hamiltonian walks on the square lattice. Extending the finite-size scaling analysis of the correlation lengths for self-avoiding walks on the square lattice [6] down to the zero-temperature limit  $t = 0$ , we see a clear convergence of the central charge and leading scaling dimension to the values  $c = -1$  and  $X_1 = 0$  for even system sizes, with  $X_2 = 0$  exactly. These results are the analog of the present study on the honeycomb lattice where  $N = 3k$  is most natural in terms of the Bethe ansatz solution.

Our results indicate that fully packed self-avoiding walks on the honeycomb lattice have the same degree of "solvability" as self-avoiding walks on the honeycomb lattice. The fully packed loop model with open boundaries is also exactly solvable [32]. The derivation of the surface critical behavior of Hamiltonian walks is currently in progress.

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