Magnetic Vortices from a Nonlinear Sigma Model with Local Symmetry

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We consider a nonlinear O(3) model in 2 + 1 dimensions minimally coupled to Chern-Simons gauge fields. All the static, finite-energy regular solutions of the model are discussed. Through a suitable reduction of the gauge group, the given solutions are mapped into an Abelian purely magnetic vortex. A two-dimensional Euclidean action reproducing such a vortex is also obtained and is that of an Abelian-Higgs model with topological term.

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The two-dimensional nonlinear O(3) model possesses solutions which are interesting both from a mathematical and physical point of view [1]. From the mathematical point of view, it provides one of the simplest nontrivial exactly solvable nonlinear systems [1–3], whereas its direct physical application is in the two-dimensional isotropic ferromagnets [1]. Moreover, the connection of this model to the long-wavelength fluctuation of antiferromagnets has been also established [4]. Beyond this, it is interesting and appealing that the model strongly resembles some crucial properties of Yang-Mills theories in four dimensions [2]. In this Letter we shall see that the nonlinear σ model (NLSM) has another unexpected application in the physics of the magnetic vortices. We elect the global O(3) symmetry of the (2 + 1)-dimensional model to be local, and we provide all the static, finite-energy regular solutions of the gauged model. These solutions are remarkably simple and reproduce, under a suitable contraction of the gauge indices, an Abelian purely magnetic vortex that corresponds to a nonperturbative solution of a two-dimensional Euclidean $|\varphi|^4$ model coupled to an Abelian gauge field.

Static soliton solutions of the (global) O(3) model in 2 + 1 dimensions are well known [1]. Given the Lagrangian [5]

$$L = \tfrac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \tfrac{1}{2}\lambda(\phi^a\phi^a - 1), \tag{1}$$

the equations of motion are

$$\phi^a\phi^a = 1, \quad \Box\phi^a = \lambda\phi^a = \phi^a\phi^b\Box\phi^b. \tag{2}$$

For static configurations, all the solutions that extremize the action satisfy the conditions [1]

$$\partial_i \phi^a = \pm\epsilon^{abc}\epsilon_{ij}\phi^b\,\partial_j\phi^c. \tag{3}$$

Moreover, each finite-energy solution satisfying (3) is characterized by a topological number

$$Q = \frac{1}{8\pi}\int d^2x\,\epsilon_{ij}\epsilon^{abc}\phi^a\,\partial_i\phi^b\,\partial_j\phi^c, \tag{4}$$

and two solutions with the same topological number $Q \in \mathrm{Z}$ are homotopically equivalent.

By performing a stereographic projection in the internal O(3) space of the sphere $\phi^a\phi^a = 1$ into the plane $\omega^1 \times \omega^2 \equiv \phi^1 \times \phi^2$, conditions (3) written in terms of ω simply become $\partial_i\omega = \pm i\epsilon_{ij}\,\partial_j\omega$, where $\omega = \omega^1 + i\omega^2$. Hence ω is an analytic (or antianalytic) function, depending on the sign chosen. For definiteness, let us consider the analytic case; then the simplest solution is for instance [6] a zero (or a pole) of order $|m|$:

$$\omega = (z/z_0)^m, \tag{5}$$

where $m \in \mathrm{Z}$, and $z_0 \neq 0$ is an arbitrary constant. The energy associated with the solution (5) is

$$E = 4\int d^2x \frac{|d\omega/dz|^2}{[1 + |\omega|^2]^2} = 4\pi|m|. \tag{6}$$

The constant z_0 is clearly related to the size of the soliton, whereas the integer m is nothing but the topological index Q of the solution so that $E = 4\pi|Q|$.

Now, we make local the global symmetry. To this purpose we introduce O(3) gauge potentials A_μ^a, a covariant derivative $D_\mu\phi^a = \partial_\mu\phi^a + \epsilon^{abc}A_\mu^b\phi^c$, and a gauge potential Lagrangian; for the latter we propose a Chern-Simons term, so that the complete Lagrangian reads

$$L = \tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a + \tfrac{1}{2}\lambda(\phi^a\phi^a - 1) - \tfrac{1}{2}\kappa\epsilon^{\mu\nu\rho}\left[\partial_\mu A_\nu^a A_\rho^a + \tfrac{1}{3}\epsilon^{abc}A_\mu^a A_\nu^b A_\rho^c\right]. \tag{7}$$

The equations of motion are then

$$\kappa F_{\mu\nu}^a = \epsilon_{\mu\nu\rho}J^{a,\rho} = \epsilon_{\mu\nu\rho}\epsilon^{abc}D^\rho\phi^b\,\phi^c, \tag{8a}$$

$$D_\mu D^\mu\phi^a = \lambda\phi^a = \phi^a\phi^b D_\mu D^\mu\phi^b, \tag{8b}$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc}A_\mu^b A_\nu^c$ is the field strength and J_μ^a the (covariantly) conserved non-Abelian matter current.

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To find static, finite-energy solutions of Eqs. (8), we minimize the energy associated with the Lagrangian (7):

$$E = \int d^2x\, \mathcal{H}_\sigma = \tfrac{1}{2}\int d^2x [D_0\phi^a D_0\phi^a + D_i\phi^a D_i\phi^a]. \tag{9}$$

Using the identity

$$\tfrac{1}{4}|D_i\phi^a \mp \epsilon^{abc}\epsilon_{ij}\phi^b D_j\phi^c|^2 = \tfrac{1}{2}D_i\phi^a D_i\phi^a \pm \tfrac{1}{2}\epsilon^{abc}\epsilon_{ij}\phi^a \partial_i\phi^b \partial_j \mp \epsilon_{ij}\,\partial_i\big(\phi^a A_j^a\big)\phi^c \pm \tfrac{1}{2}\epsilon_{ij}\phi^a F_{ij}^a,$$

and taking Eqs. (4) and (8a) into account, it follows that

$$E = \tfrac{1}{2}\int d^2x\Big[D_0\phi^a D_0\phi^a + \tfrac{1}{2}\,|D_i\phi^a \mp \epsilon^{abc}\epsilon_{ij}\phi^b D_j\phi^c|^2\Big] + 4\pi|Q| \geq 4\pi|Q|\,. \tag{10}$$

In passing from Eq. (9) to Eq. (10) we dropped the surface term $\epsilon_{ij}\int d^2x\, \partial_i(\phi^a A_j^a)$, as consistent with the asymptotic behavior of the solutions we shall provide.

Clearly, for static solutions, the minimum $E = 4\pi|Q|$ of the energy is attained when

$$(A_0^a A_0^a) = (A_0^a\phi^a)^2, \tag{11a}$$

$$D_i\phi^a = \pm\epsilon^{abc}\epsilon_{ij}\phi^b D_j\phi^c. \tag{11b}$$

It is easy to check that conditions (11) are indeed equivalent to Eq. (8b), the configuration being static. Equation (11a) fixes the gauge structure of A_0^a as $A_0^a = c(x)\phi^a$, and the value of $c(x)$ can be obtained by consistency between Eqs. (8a) and (11b), which gives $c = \mp(1/\kappa)$. The remaining components A_i^a of the gauge potentials are pure gauges, as can be seen by substituting $A_0^a = \mp(1/\kappa)\phi^a$ in Eq. (8a). Therefore, all the static, finite-energy configurations that extremize the action of the model (7) satisfy the conditions

$$D_i\phi^a = \pm\epsilon^{abc}\epsilon_{ij}\phi^b D_j\phi^c, \tag{12a}$$

$$A_i^a = i\,\mathrm{Tr}\,(\sigma^a U^{-1}\partial_i U), \tag{12b}$$

$$A_0^a = \mp(1/\kappa)\,\phi^a, \tag{12c}$$

for any gauge group element $U = \exp[-i\alpha^a(x)\sigma^a/2]$, σ^a being the Pauli matrices. Equations (12a) and (12b) imply that the matter field solutions of the gauged NLSM are trivially equivalent to those of the nongauged NLSM [trivial part of Eqs. (12)]. In fact, being Eqs. (12) are covariant under gauge transformations, Eq. (12a) can be replaced by Eq. (3) when written in terms of the U-gauge transformed field $\phi \rightarrow U^{-1}\phi U$. Consequently, making local the O(3) symmetry does not affect either the value of the energy or the matter field solutions: Any analytic (or antianalytic) function $\omega(z)$ in the stereographically projected plane is a solution of Eq. (12a). The corresponding value of A_0^a [nontrivial part of Eqs. (12)] can be immediately obtained from Eq. (12c).

These non-Abelian solutions give rise to a nontrivial vortex configuration under a suitable contraction of the gauge group: In fact, we can construct a gauge invariant Abelian field strength by setting [7,8]

$$\mathcal{F}_{\mu\nu} = \epsilon^{abc} D_\mu\phi^a D_\nu\phi^b\phi^c - \phi^a F_{\mu\nu}^a. \tag{13}$$

By substituting Eqs. (12) in (13), the Abelian field strength has a vanishing electric field $\mathbf{E} = 0$ and a magnetic field which is equal (up to a sign) to the non-Abelian Hamiltonian $\mathcal{H}_\sigma$,

$$B = \pm\mathcal{H}_\sigma = \pm\frac{4|\omega'|^2}{[1+|\omega|^2]^2}, \tag{14}$$

where $\omega' = d\omega(z)/dz$ [or $d\omega(\bar{z})/d\bar{z}$] for the choice of sign + (or −). This is the typical configuration of a purely magnetic Abelian vortex. In fact, the flux of the magnetic field is quantized

$$\Phi = \int d^2x\, B = \pm\int d^2x\,\mathcal{H}_\sigma = \pm 4\pi|Q|, \tag{15}$$

Q being the integer characterizing the corresponding non-Abelian solution [see Eq. (4)].

For definiteness, from now on we shall choose $\omega(z)$ analytic as in Eq. (5). Then, the sign in Eqs. (14) and (15) is +, $\Phi = 4\pi|m|$, and

$$B = \frac{4m^2}{r^2}\left[\left(\frac{r}{r_0}\right)^m + \left(\frac{r_0}{r}\right)^m\right]^{-2}, \tag{16}$$

where $|z_0| = r_0 \neq 0$.

Since $\mathbf{E} = 0$, A^0 can be chosen to vanish, the configuration being static, whereas an Abelian gauge potential $\mathbf{A}$ reproducing the magnetic field (16) is, for instance,

$$\mathbf{A}(\mathbf{r}) = \frac{2|m|}{r}\left[1 + \left(\frac{r_0}{r}\right)^{2|m|}\right]^{-1}\mathbf{e}_\theta. \tag{17}$$

It should be noticed that the profile of the magnetic field is identical to that of the soliton solutions of the gauged nonlinear Schrödinger equation recently discussed by Jackiw and Pi [9]. However, in our case the vortex is electrically neutral, whereas in the Jackiw-Pi solitons, as the dynamics of the gauge fields is governed by an Abelian Chern-Simons term, any excitation with magnetic flux Φ necessarily has a nonvanishing electric field [9].

The identification (13) was first proposed in Ref. [7], in a (3 + 1)-dimensional context, reproducing magnetic monopole solutions from static solutions of an SU(2) Yang-Mills-Higgs model [7,10]. In our (2 + 1)-dimensional case, the same identification (13) yields to purely magnetic vortices from the solutions of the NLSM. However, an important difference between the contraction (13) in 3 + 1 and lower dimensions should be pointed out. In 3 + 1 dimensions, the field strength $\mathcal{F}_{\mu\nu}$ defined as in (13) and reproducing the monopoles does not satisfy the sourceless Maxwell equations (Bianchi identities) $\epsilon^{\mu\nu\rho\sigma}\partial_\nu\mathcal{F}_{\rho\sigma} = 0$. Rather, the divergence of the dual tensor ${}^*\mathcal{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}$ defines the conserved "magnetic" current, whose charge is just the monopole charge. On the contrary, in 2 + 1 dimensions, the field strength $\mathcal{F}_{\mu\nu}$ given in (13) does indeed satisfy the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu\mathcal{F}_{\nu\rho} = 0$. Consequently, in 2 + 1 dimensions one might wonder whether an Abelian model reproducing the magnetic vortices (17) exists. In the affirmative case, the vortices would be completely independent from the NLSM, contrary to what happens in the corresponding (3 + 1)-dimensional case with the monopoles and the SU(2) Yang-Mills-Higgs solitons.

An Abelian model that reproduces the magnetic field (16) is, as already mentioned, the Jackiw-Pi soliton [9]. Here, we shall provide a different model reproducing the same magnetic field.

Since all the temporal components of the Abelian configuration (13) trivially vanish (static solutions with $\mathbf{E} = 0$), instead of looking for static solutions of a (2 + 1)-dimensional model, we shall consider a Euclidean two-dimensional model reproducing the nontrivial (magnetic) part of the vortex. We propose a complex scalar field minimally coupled to a Maxwell gauge potential, with an additional topological coupling:

$$\mathcal{L} = \tfrac{1}{4}\mathcal{F}_{ij}\mathcal{F}_{ij} + (D_i\varphi)^* D_i\varphi - \varphi^*\varphi\epsilon_{ij}\mathcal{F}_{ij} + V(\varphi^*\varphi), \tag{18}$$

where $D_i\varphi = \partial_i\varphi - iA_i\varphi$ and $V(\varphi^*\varphi)$ is a scalar potential to be determined. The Lagrangian (18) is both ISO(2) invariant as well as U(1) gauge invariant. Substituting the gauge potential (17) in the Maxwell equations arising from (18)

$$\partial_i\mathcal{F}_{ij} - 2\epsilon_{ij}\partial_i(\varphi^*\varphi) = i(\varphi^* D_j\varphi - \varphi D_j\varphi^*) \equiv J_j, \tag{19}$$

one can get the form of the scalar field φ such that Eq. (19) is satisfied. Up to an irrelevant constant phase, it reads

$$\varphi(\mathbf{r}) = \frac{2|m|}{r}\left[\left(\frac{r}{r_0}\right)^m + \left(\frac{r_0}{r}\right)^m\right]^{-1} \mathbf{e}^{i(|m|-1)\theta}. \tag{20}$$

The potential $V(\varphi^*\varphi)$ is determined by imposing that the scalar field $\varphi(\mathbf{r})$ given in Eq. (20) satisfies its own equation of motion, $[\mathcal{D}_i\mathcal{D}_i + \epsilon_{ij}\mathcal{F}_{ij} - dV/d(\varphi^*\varphi)]\varphi = 0$, and solving this equation for $V(\varphi^*\varphi)$. The solution is remarkably simple and gives

$$V(\varphi^*\varphi) = \tfrac{1}{2}(\varphi^*\varphi)^2. \tag{21}$$

From Eqs. (17) and (20), both $\varphi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are always regular if $|m| \geq 1$. Asymptotically, $|\varphi(\mathbf{r})| \sim r^{-|m|-1}$, whereas $\mathbf{A} \sim 2m\nabla\theta$, which is the correct behavior reproducing $\Phi = 4\pi|m|$. Therefore, for any $m \in Z$ and different from zero, Eqs. (17) and (20) provide nonperturbative, regular solutions of the Lagrangian (18) with potential (21).

Stability of these solutions can be established by substituting the identity

$$D_i\varphi^* D_i\varphi = |(D_x \pm iD_y)\varphi|^2 \pm (\varphi^*\varphi)B \pm \tfrac{1}{2}\epsilon_{ij}\partial_i J_j, \tag{22}$$

in the classical action, obtaining

$$S = \int d^2x\,\mathcal{L} = \int d^2x\left[|(D_x \pm iD_y)\varphi|^2 + \tfrac{1}{2}(B \mp \varphi^*\varphi)^2\right] \geq 0, \tag{23}$$

where we omitted the last (surface) term in Eq. (22) as it vanishes when integrated. From Eqs. (16), (17), and (20) it can be easily seen that the solutions satisfy the following conditions:

$$B = \varphi^*\varphi, \quad D_i\varphi = -i\epsilon_{ij}D_j\varphi. \tag{24}$$

[Had we chosen the minus sign in Eqs. (14) and (15), the right-hand side of conditions (24) would have changed sign.] Consequently, on the classical solutions, the Euclidean action achieves its minimum $S = 0$. As expected, the same conditions (24) are also satisfied by the Jackiw-Pi solitons [9], although in a different context and with a different Lagrangian. The second condition (24) shows that the Abelian solutions are self-dual: the Abelian contraction (13) maps O(3) self-dual solutions in U(1) self-dual ones. Combining together conditions (24), one finds that $\varphi^*\varphi$ satisfies the Liouville equation,

$$\Delta\ln(\varphi^*\varphi) = -2(\varphi^*\varphi). \tag{25}$$

This is not surprising: $|B|$ in Eq. (14) provides, in fact, the most general solution of the Liouville equation (25), whereas Eq. (16) gives its radially symmetric solution.

Here, we conclude with two comments:

(a) The generalization of the Abelian solutions when $\omega(z)$ is an arbitrary analytic function is straightforward and gives

$$A_i = -\epsilon_{ij}\,\partial_j \ln[1 + |\omega(z)|^2]\,, \quad \varphi = \frac{2\omega'(z)}{1 + |\omega(z)|^2}\,. \quad (26)$$

These are the general solutions of Eqs. (24). The radially symmetric solutions previously discussed are obtained by choosing $\omega(z)$ as in (5). N-vortex solutions of the Abelian model can be easily obtained, for a specific choice of $\omega(z)$, following the same lines of Ref. [9]. Such solutions are very likely related to the N-soliton solutions of the NLSM under contraction (13) of the O(3) group.

(b) The presence of the topological coupling $(\varphi^*\varphi)\epsilon_{ij}\mathcal{F}_{ij}$ in the Euclidean action makes it difficult to figure out a possible $(2 + 1)$-dimensional action, reproducing Eq. (19) as its static Hamiltonian. Rather, an intriguing possibility for future investigations could be to interpret the Lagrangian (18) as an effective Lagrangian of a more elementary theory. To this purpose, it should be noticed that the topological coupling could be generated by bosonization of fermions minimally coupled to the gauge field [11].

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