Entropy in Dilatonic Black Hole Background

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The entropy of a scalar field is calculated semiclassically in the background of a dilatonic black hole. The area and cutoff dependences are normal *except in the extremal case,* where the area is zero but the entropy nonzero.

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It is well known that the area of the horizon of a black hole can be interpreted as an entropy [1] and satisfies all the thermodynamical laws. This is not yet understood in terms of the usual formulation of entropy as a measure of the number of states available, but the naïve Lagrangian path integral does lead to a partition function from which the area formula for entropy can be obtained [2] by neglecting quantum fluctuations.

There have also been some attempts at calculating the entropy of quantum fields in black hole backgrounds [3,4]. The values thus obtained are contributions to the entropy of the black-hole–field system. These calculations have produced divergences, but the area of the horizon has appeared as a factor. This has been interpreted to mean that the gravitational constant gets renormalized in the presence of the quantum fields [4]. We shall investigate whether similar phenomena occur in the case of *dilatonic* black holes [5,6] where it is possible to have a *vanishing* horizon area.

The low-energy limit of string theory with unbroken supersymmetry contains a massless dilaton field. Models where these dilatons are coupled with gravity may be used for studying black holes with small Compton wavelengths. The simplest four-dimensional model is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,e^{-2\phi}(R + 4g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi), \tag{1}$$

where ϕ is the massless dilaton field, R is the scalar curvature, and $g_{\mu\nu}$ is the metric. As the curvature term contains an extra exponential factor, this is often removed by the conformal transformation

$$\tilde{g}_{\mu\nu} = e^{-2\phi}g_{\mu\nu}. \tag{2}$$

This changes the action to

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\,(\tilde{R} - 2\tilde{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi), \tag{3}$$

which is standard Einstein gravity coupled with a massless scalar field. Thus $\tilde{g}_{\mu\nu}$ is the appropriate metric for gravitational studies and in fact all theorems of general relativity are applicable in this metric. This is not the case with the original string metric $g_{\mu\nu}$, which, however, is the metric seen by the string. We shall confine ourselves to the metric $\tilde{g}_{\mu\nu}$.

The model can be extended to have electromagnetic interactions by including the term

$$-\frac{1}{32\pi}\int d^4x\sqrt{-\tilde{g}}\,e^{-2\phi}\tilde{g}^{\mu\lambda}\tilde{g}^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} \tag{4}$$

in the action (3). Exact black hole solutions of this model have been found with nonzero charge and angular momentum.

The black hole solution with zero angular momentum strongly resembles the Schwarzschild solution of standard general relativity:

$$\begin{aligned} d\tilde{s} &= \tilde{g}_{\mu\nu}dx^\mu dx^\nu \\ &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 \\ &\quad + r(r-a)\,d\Omega_{II}^2, \\ e^{-2\phi} &= e^{-2\phi_0}\left(1 - \frac{a}{r}\right), \\ F_{\theta\varphi} &= Q\sin\theta, \end{aligned} \tag{5}$$

where M is the mass of the black hole, Q its magnetic charge, the parameter a is defined by

$$a = \frac{Q^2}{2M}e^{-2\phi_0}, \tag{6}$$

and ϕ_0 is an arbitrary constant. This black hole has, as usual, a horizon at $r = 2M$. An interesting feature is that a curvature singularity occurs at $r = a$. The so-called extremal solution corresponds to the coincidence of these two regions and thus has $a = 2M$. This extremal limit is interesting also because the area $4\pi 2M(2M - a)$ of the horizon vanishes. All this is from the point of view of the gravitational metric. However, from the string theory point of view, the geometry in the extremal limit is perfectly nonsingular. In the string metric the horizon disappears and as $r \to 2M$, the spacetime splits into a $(1+1)$ dimensional Minkowski spacetime times a sphere of constant radius $2M$ (the *throat*).

As argued in [7], the partition function for the system can be defined by the (Euclidean) Lagrangian path integral for the gravitational action coupled with matter fields. The dominant contribution will come from the classical solutions of the action. We may approximate the Euclidean action by taking something like

$$S_E[\tilde{g},\phi,F,\varphi] = S_1[\tilde{g}_{cl},\phi_{cl},F_{cl}] + S_2[\tilde{g}_{cl},\varphi] + \cdots, \tag{7}$$

where φ is the scaler field to be considered in the background of the dilatonic black hole. Quantum fluctuations of the metric, the electromagnetic field and the dilatonic field, are neglected and these variables are frozen to their classical values. The partition function can then be taken as

$$Z = e^{-S_1[\tilde{g}_{cl},\phi_{cl},F_{cl}]}\int [d\phi]\, e^{-S_2[\tilde{g}_{cl},\varphi]}. \tag{8}$$

The contribution of the piece S_1 to the entropy of the dilatonic black hole is known to be given by one-fourth of the area of the horizon [8]. We consider the contribution of the scalar field φ to the partition function through the piece S_2.

We employ the brick-wall boundary condition [3]. In this model the wave function is cut off just outside the horizon. Mathematically,

$$\varphi(x) = 0 \quad \text{at } r = 2M + \epsilon, \tag{9}$$

where ϵ is a small, positive, quantity and signifies an ultraviolet cutoff. There is also an infrared cutoff

$$\varphi(x) = 0 \quad \text{at } r = L, \tag{10}$$

with $L \gg 2M$.

The wave equation for a scalar field in this spacetime reads

$$\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - m^2\varphi = 0. \tag{11}$$

A solution of the form

$$\varphi = e^{-iEt} f_{El} Y_{lm_l} \tag{12}$$

satisfies the radial equation

$$\left(1-\frac{2M}{r}\right)^{-1} E^2 f_{El} + \frac{1}{r(r-a)}\frac{\partial}{\partial r}\left[(r-a)(r-2M)\frac{\partial f_{El}}{\partial r}\right] - \left[\frac{l(l+1)}{r(r-a)} + m^2\right] f_{El} = 0. \tag{13}$$

An r-dependent radial wave number can be introduced from this equation by

$$k^2(r,l,E) = \left(1-\frac{2M}{r}\right)^{-1}\left[\left(1-\frac{2M}{r}\right)^{-1} E^2 - \frac{l(l+1)}{r(r-a)} - m^2\right]. \tag{14}$$

Only such values of E are to be considered here that the above expression is non-negative. The values are further restricted by the semiclassical quantization condition

$$n_r\pi = \int_{2M+\epsilon}^{L} dr\, k(r,l,E), \tag{15}$$

where n_r has to be a non-negative integer.

Accordingly, the free energy F at inverse temperature β is given by the formula

$$\begin{aligned}
\beta F &= \sum_{n_r,l,m_l} \ln(1-e^{-\beta E}) \approx \int dl\,(2l+1)\int dn_r \ln(1-e^{-\beta E}) \\
&= -\int dl\,(2l+1)\int d(\beta E)\,(e^{\beta E}-1)^{-1} n_r \\
&= -\frac{\beta}{\pi}\int dl\,(2l+1)\int dE\,(e^{\beta E}-1)^{-1}\int_{2M+\epsilon}^{L} dr\left(1-\frac{2M}{r}\right)^{-1}\sqrt{E^2-\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r(r-a)}+m^2\right)} \\
&= -\frac{2\beta}{3\pi}\int_{2M+\epsilon}^{L} dr\left(1-\frac{2M}{r}\right)^{-2} r(r-a)\int dE\,(e^{\beta E}-1)^{-1}\left[E^2-\left(1-\frac{2M}{r}\right)m^2\right]^{3/2}.
\end{aligned} \tag{16}$$

Here the limits of integration for l, E are such that the arguments of the square roots are non-negative. The l integration is straightforward and has been explicitly carried out. The E integral can be evaluated only approximately.

The contribution to the r integral from large values of r yields the expression for the free energy valid in flat spacetime ($M = 0$):

$$F_0 = -\frac{2}{9\pi} L^3 \int_m^\infty dE \, \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1} . \tag{17}$$

We ignore this part [2,3]. The contribution of a nonzero M is singular in the limit $\epsilon \to 0$. The leading singularity is linear:

$$F_{\text{lin}} \approx -\frac{2\pi^3}{45\epsilon}\left(1 - \frac{a}{2M}\right)\left(\frac{2M}{\beta}\right)^4, \tag{18}$$

where the lower limit of the E integral has been approximately set equal to zero. If the proper value is taken, there are corrections involving $m^2\beta^2$, which will be ignored here. This result reduces to the formula [3] for the Schwarzschild black hole when $a = 0$. In general, there is simply a multiplicative factor $(1 - a/2M)$.

There is a logarithmic singularity as well, but it is in general ignored because of the presence of the linearly divergent term F_{lin}. However, the linear term vanishes when $a = 2M$, i.e., when the black hole becomes extremal. In this case, the logarithmic term is the dominant one. It is

$$F_{\log} \approx -\frac{\pi^3}{45M} \ln\left(\frac{2M}{\epsilon}\right)\left(\frac{2M}{\beta}\right)^4 \tag{19}$$

in the same approximation as above.

The entropy due to a nonzero M can be obtained from the formula

$$S = \beta^2 \frac{\partial F}{\partial \beta}. \tag{20}$$

This gives

$$S = \frac{8\pi^3}{45}\left(\frac{2M}{\beta}\right)^3 \frac{(1 - \frac{a}{2M})\,(2M)^2}{2M\epsilon} \quad \text{for } a \neq 2M , \tag{21}$$

and

$$S = \frac{8\pi^3}{45}\left(\frac{2M}{\beta}\right)^3 \ln\frac{(2M)^2}{2M\epsilon} \quad \text{for } a = 2M . \tag{22}$$

Thus, for $a \neq 2M$, namely for nonextremal dilatonic black holes, the Schwarzschild expression is valid, but with the area factor $(2M)^2$ corrected by the appropriate coefficient $(1 - a/2M)$. Note that the factor $(2M/\beta)$ is a constant if the Hawking temperature is used, because of its inverse dependence on the mass, while the quantity $2M\epsilon$ may be regarded as giving an invariant measure of the distance of the brick wall from the horizon [3]. As the entropy of the dilatonic black hole itself is $S =$ (area)/$(4G)$ (where G has been set equal to unity) [8], the above divergent contribution may be understood as a renormalization of the gravitational coupling constant G [4]. However, quantum gravity being nonrenormalizable, this interpretation cannot be extended to include quantum fluctuations of the gravitational fields.

In the case of *extremal* dilatonic black holes a logarithmic formula appears, in which the usual factor $(2M)^2/2M\epsilon$ is *replaced by its logarithm.* For these black holes, where the area of the horizon vanishes, one might have expected the entropy to vanish altogether. What does happen is that the linear divergence vanishes, but the logarithmic divergence, which is of course weaker, stays on. A similar logarithmic divergence is known to occur if the theory is truncated to $(1 + 1)$ dimensions [4]. Our calculation shows that this is already present in $(3 + 1)$ dimensions. But this divergence cannot be regarded as a renormalization of G because even with a renormalized G, the zero area should have made the entropy zero.

To summarize, we have calculated semiclassically the entropy of a scalar field in the background of a dilatonic black hole. The results are similar to the case of ordinary black holes and involve a linearly divergent renormalization in general. But the area of the horizon may vanish here, and in that extremal case, the entropy of the scalar field does not vanish. The singularity becomes logarithmic.

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