Line Dispersion in Homogeneous Turbulence: Stretching, Fractal Dimensions, and Micromixing

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Motivated by the understanding of the turbulent mixing mechanisms at small scale, the time evolution of an initially regular passive scalar pattern in three-dimensional homogeneous turbulence is investigated in a line-dispresion experiment. The major observation is the increasing space-fillingness of the line as time elapses. It is shown that the evolution of the fractal dimension $d_f(t)$ of the support of the line can be related to the mean-field Kolmogorov scaling of velocity differences incorporating dissipative corrections and an expression for $d_f(t)$ is derived that compares very well with the experiments for several distinct Reynolds numbers. Following, a criterion of micromixing is discussed.

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The ability to distort material fluid surfaces in complex convoluted geometries is one of the most common and striking features of turbulent flows. In most high Reynolds number shear flows, the growth of the primary instabilities results in the fragmentation of the streams involved in macroscopic "coherent packets." A "packet," whose frontier is initially smooth is further stretched and folded by the smaller-scale activity of the flow to ultimately fill the available space in a more or less compact manner. This is the problem of turbulent mixing which is particularly important when chemical reactions occur between the mixed streams since the stretching of the turbulent motions and the consecutive area generation of the frontier of the packet govern the amount of area per unit of volume of the interface and the intensity of the exchanges across it.

In order to address this question in the most general terms, we have considered the case for which the "frontier" has initially the simplest shape of a straight line [1, 2], boarding a ribbon of passive tracer immersed at the origin of time in a grid-generated turbulence.

Our goal is to depict the evolution of the geometry of the support of the line [3] as a function of time and to understand how the scale-dependent stretching in the flow governs the time-dependent roughness or "space-fillingness" of the line.

The experiments have been conducted in the 80×80 cm^2 vein of a wind tunnel. A 1/10 mm tungsten wire was stretched across the whole section height normal to the mean flow 30 meshes downstream of a square mesh ($M = 7$ cm) square rod (1.5 cm thick) grid. The ribbon of tracer is realized by the vaporization of a thin liquid oil film coated on the wire. The extremities of the wire are connected to a condensator which discharges in about 30 ms under 60 V through the wire. The vaporization time of the oil film is nevertheless larger than the wire heating time, increased by the cooling time during which the surface temperature of the wire is above the oil vaporization temperature. Once shedded, the smoke ribbon produced is convected downstream, losing progressively its rectilinear shape under the action of the turbulent velocity field (the Péclet number is much larger than unity). It is made visible by a white homogeneous light sheet, sufficiently thick to contain the whole distorted ribbon at all times. The temporal evolution of the ribbon is recorded, through the transparent walls of the wind tunnel, by a charge-coupled device video camera positioned at right angle with respect to the direction of the mean flow at the rate of 25 frames per second. The images are further processed, one by one, in the following way: on each digitized picture (768×512 pixels on 256 grey levels), the ribbon is extracted from the background and its external frontier is particularized (Fig. 1). One then has access that way to the concentration field of smoke along the ribbon and to its external frontier shape as a function of time by steps of $\frac{1}{25}$ s.

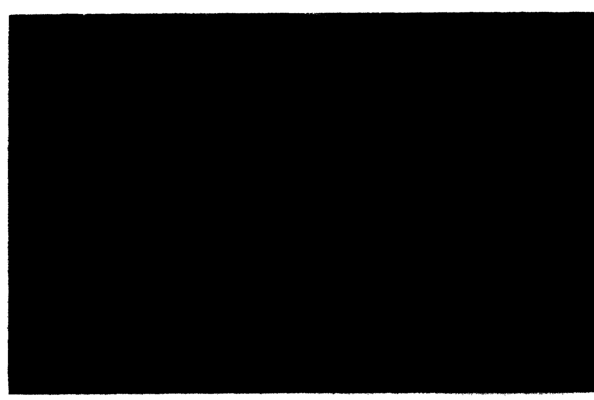

FIG. 1. Evolution, at different consecutive instants spaced by steps of 2/25 sec, of an advected tracer ribbon in a grid-generated turbulence. Re = 33. The external frontier of the ribbon is particularized by the white contour. The integral length scale L is 2.2 cm and the height of the picture covers $30L$.

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This line defines the support of the ribbon, independently of the different local concentration levels which, as for them, reflect the cumulated stretchings the ribbon has undergone. As stated above, we want to characterize the support of the line by a parameter which reflects its "roughness," its space-fillingness, and the hierarchy of spatial scales that participate to its deformation. As a function of time, and for different moderate Reynolds numbers, we have measured the fractal dimension of the projected line. One can show [4, 5] that the fractal dimension of the projection of the line is equal to the dimension of the actual line embedded in the three-dimensional space as long as this dimension remains less than 2; this is always the case in the present experiments. In grid turbulence, the root mean square velocity fluctuation u' and the turbulence integral length scale L depend on the mean velocity U, on the mesh size M of the grid, and on the downstream location. They have been computed from the laws given by Batchelor and Comte-Bellot and Corrsin [6]. The Kolmogorov length scale η is about 0.15 cm, L is about 2.3 cm, and the Taylor scale λ is about 1.5 cm. The Reynolds number $\mathrm{Re} = u'L/\nu$ varied from 18 to 35 ($\mathrm{Re}_M = UM/\nu$ about 7200). The resolution of the images was 11 pixels per centimeter in such a way that an image typically covers 30 integral scales L and that η is just resolved.

The central observation is that the fractal dimension of the line, which is initially 1 (straight line), actually increases with time. The increase is linear and all the more rapid that the Reynolds number is large [Figs. 2(a) and 2(b)]. These results are meaningful for short time intervals since the local features of the turbulence gradually change downstream of the grid. The turbulent kinetic energy u'^2 behaves like $t^{-\alpha}$ with $\alpha \approx 1.35$ and L like t^{β} with $\beta \approx 0.4$ [6]. Although these are power-laws, the turbulent field in which the line is embedded cannot be considered as stationary for time intervals larger than the local integral scale turnover time $t(L) = L/u'$ but, as it is shown now, this point is of negligible importance.

Since Kolmogorov 1941 [7], the attention has been drawn to the longitudinal velocity difference $\delta u(r)$ taken between two points separated by a distance r. When the energy is assumed to be dissipated at a uniform rate ε in the medium, a direct dimensional argument yields

$$\delta u(r) \sim (\varepsilon r)^{\zeta} \tag{1}$$

with $\zeta = \frac{1}{3}$. In this expression, the notation $\delta u(r)$ represents the rms value of the longitudinal velocity difference: $\sqrt{\langle \delta u^2(r) \rangle}$. The law (1), and its r dependence has been extensively measured with a good accuracy (except for a very weak intermittency correction at this structure function order) and can be considered, at large Reynolds numbers, as an experimental fact [8]. The relation (1) gives the separation velocity of two particles or, equivalently, the stretching velocity of the segment which links them for scales r lying in the inertial range ($\eta < r < L$). The relative diffusion of two particles

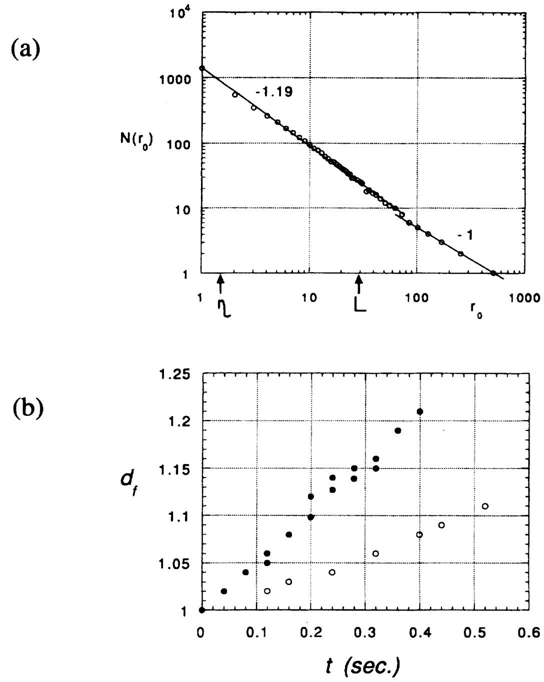

FIG. 2. (a) Log-log plot of the number $N(r_0)$ of segments of size r_0 needed to cover a contour line, as a function of r_0, counted in pixels. η and L are respectively the Kolmogorov and integral length scale. $\mathrm{Re} = 33$, $t/t(L) = 0.375$. (b) Fractal dimension of the lines as a function of time, for two distinct Reynolds numbers. •, $\mathrm{Re} = 33$, $t(L) = 0.97$ sec; ∘, $\mathrm{Re} = 18$, $t(L) = 1.92$ sec.

is an increasing function of r, from which follows the celebrated Richardson "four-third law" [9]. Nearer from the dissipation scale η, the velocity difference $\delta u(r)$ goes to zero proportionally to r: viscous damping only allows for solid rotation or simple shear. It is known that, for small τ separations, $\delta u(r)^2 \sim u'^2(r^2/\lambda^2)$ and thus, with $\eta = (\nu^3/\varepsilon)^{1/4}$, that the velocity difference itself writes $\delta u(r) \overset{r\to 0}{\longrightarrow} (r/\eta)^{1-1/3} (\varepsilon r)^{1/3}$ which indeed goes to zero like r for $r \to 0$ [10]. We want to describe continuously the r dependence of the velocity $\delta u(r)$ from the dissipative scale η to the inertial range. We thus look for a function $f(r/\eta)$ such that

$$\delta u(r) = f(r/\eta)\,(\varepsilon r)^{\zeta} \tag{2}$$

with the following constraints

$$f(r/\eta) \overset{r/\eta \gg 1}{\longrightarrow} 1, \tag{3a}$$

$$f(r/\eta) \overset{r/\eta < 1}{\longrightarrow} \left(\frac{r}{\eta}\right)^{1-\zeta}. \tag{3b}$$

A simple form for the continuous function f may be chosen as

$$f(r/\eta) = 1 - e^{-(r/\eta)^{1-\zeta}} \tag{4}$$

which expresses the relaxation from the scaling law (1) for $r/\eta \gg 1$ to the viscous dominated régime for $r/\eta \gg 1$. The slow transition through the dissipative range is expressed by the slow relaxation (4) in such a way that, for moderate Re, the velocity difference (2) is smoother and much less singular than (1). Such a crossover function has been successfully introduced by Benzi *et al.* [11] to extend the similarity scaling régime of structure functions to the dissipative range. The function f can be exactly computed from the Kolmogorov equation

$$\langle \delta u^3(r)\rangle = -\tfrac{4}{5}\varepsilon r + 6v\frac{d\langle \delta u^2(r)\rangle}{dr}$$

and one can show that the expression $f(r/\eta) = \mathrm{Erf}\left[\sqrt{\pi b}\,(r/\eta)^{2/3}/2\right]$ with $b = (4/5)^{1/3}/12$ is a very good approximation of the real solution (the expansion of the exact solution and the one of this function coincide up to the second order in r and their maximal relative discrepancy does not exceed 4% on the whole r/η range). The form of $f(r/\eta)$ chosen here Eq. (4) is only meant to ensure the scaling constraints (3a) and (3b) and is sufficiently simple to be easily analytically tractable [12]. Another approximate analytical expression has been proposed recently [13]. We insist on the fact that the precise form for $f(r/\eta)$ is of little importance for the present discussion. We further assume that the law for the velocity differences given by (2) and (4) holds everywhere in the medium and that (i) deviation in scale (intermittency) and (ii) deviation in intensity [existence of a continuous probability density function for $\delta u(r)$] are irrelevant [14].

Let us come back to the problem of the line, immersed in a three-dimensional "à la Kolmogorov" turbulence. The starting point of our analysis consists to represent the line (of initial length L_0) as a collection of $N_0(r_0) = L_0/r_0$ segments of size r_0 jointed end to end (Fig. 3). After a short time interval τ that we subsequently specify, each segment has seen, in the mean, its length increased by

$$r(\tau) = r_0 + \delta u(r_0)\tau\,. \tag{5}$$

At that time τ, the length of the line is $L(\tau) = N_0(r_0)r(\tau)$ and is constituted by

$$N_\tau(r_0) = \frac{L(\tau)}{r_0} = \frac{L_0}{r_0}\left(1 + f(r_0/\eta)\frac{\tau\varepsilon^\zeta}{r_0^{1-\zeta}}\right) \tag{6}$$

segments of size r_0. The above equation displays a central result: One sees, at the first step of the construction of the arborescence of the line, where the fractality comes from. The amplification factor $[1 + f(r_0/\eta)(\tau\varepsilon^\zeta/r_0^{1-\zeta})]$ is a decreasing function of the scale r_0 because of the factor $1/r_0^{1-\zeta}$ (remember that $\zeta < 1$). It is thus all the more large that r_0 is small: the small scales reproduce themselves faster than the large scales [they have a shorter turnover time $r_0/\delta u(r_0)$] and thus, after a given lapse of time, their number is proportionally larger than the number of large scales (within the trivial $1/r_0$ factor of course). One might thus expect that the fractal dimension of the line $d_f(t)$ defined by

$$N_t(r_0) \sim r_0^{-d_f(t)} \tag{7}$$

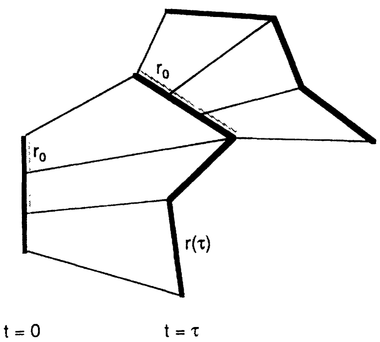

FIG. 3. The first step of the arborescent pair diffusion process, at a given scale r_0.

is an increasing function of time t, thus relating the faster multiplication of the small scales. Incidently, this reasoning is based on the fact that $\zeta = \frac{1}{3} < 1$. Indeed, for $\zeta = 1$, the number of scales of size r_0 contained by the line increases by a factor independent of r_0 [Eq. (6)].

Pursuing the arborescence step by step, the number $N_t(r_0)$ of segments of size r_0 in the line at time t is

$$N_t(r_0) = N_0(r_0)\left[1 + f(r_0/\eta)\frac{\tau\varepsilon^\zeta}{r_0^{1-\zeta}}\right]^{t/\tau}. \tag{8}$$

The time-interval τ is the "clock" of this multiplicative process and corresponds to the smallest physical time scale of the flow, that is to the turnover time of the smallest scale η of the medium

$$\tau \sim \frac{\eta}{\delta u(\eta)} = \frac{\eta^{1-\zeta}}{f(1)\,\varepsilon^\zeta}. \tag{9}$$

Setting $y = r_0/\eta$, with $\eta/L = \mathrm{Re}^{-1/(1+\zeta)}$ and $t(L) = L/(\varepsilon L)^\zeta$, (8) gives

$$N_t(y) = \frac{\mathrm{Re}^{1/(1+\zeta)}}{y} \times\left[1 + \frac{f(y)}{f(1)\,y^{1-\zeta}}\right]^{[t/t(L)]f(1)\,\mathrm{Re}[(1-\zeta)/(1+\zeta)]}. \tag{10}$$

The dependence $N_t(y)$ versus y is a quasi power law at finite Reynolds number and for $t/t(L)$ smaller than unity (Fig. 4). The apparent fractal dimension of the line is the slope at $y = 1$ (that is $r_0 \sim \eta$) in log-log coordinates

$$d_f = -\left(\frac{d\log N_t(y)}{d\log y}\right)_{y=1}, \tag{11}$$

and is found to be

$$d_f = 1 + (1-\zeta)\left[\frac{1 - 2/e}{2}\right]\mathrm{Re}^{[(1-\zeta)/(1+\zeta)]}\frac{t}{t(L)}. \tag{12}$$

According to the remark made above, if ζ is equal to unity, then $d_f(t)$ remains equal to 1 at all times t. With $\zeta = \frac{1}{3}$, (12) yields

$$d_f = 1 + 0.088\frac{t}{t(L)}\mathrm{Re}^{1/2}. \tag{13}$$

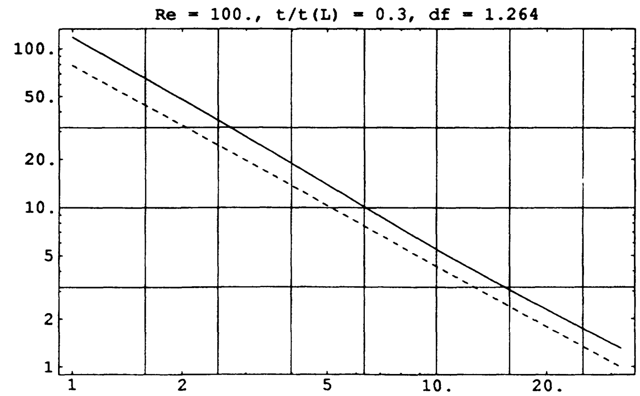

FIG. 4. *Continuous line*: $N(y)$ versus y given by Eq. (10); $y = r_0/\eta$; *Dashed line*: Corresponding power law [Eq. (7) and (13)] $N_t(y) \sim y^{-d_f(t)}$. Re = 100. $t/t(L) = 0.3$.

The fractal dimension increases linearly with time, the faster the larger the Reynolds number. The length of the line is predicted by Eq. (13) to grow exponentially with time $[L_t(y) \sim y^{1-d_f(t)}]$ at a rate proportional to $\mathrm{Re}^{1/2}/t(L)$ [15]. Our experimental results fully agree with this law (Fig. 5). Although it is restricted to short times because the underlying mechanism deserves an arborescent pair diffusion process, the validity of (13) is not ruled out by our experiments which, however, extend up to $t/t(L) \approx 0.4$. As soon as the line is sufficiently folded back on itself with, say, "hair pin" foldings of width η, then the construction sketched in Fig. 3 is not possible any more. At this critical time t_M (Micromixing time), the small structures of size η start to fill the three-dimensional space

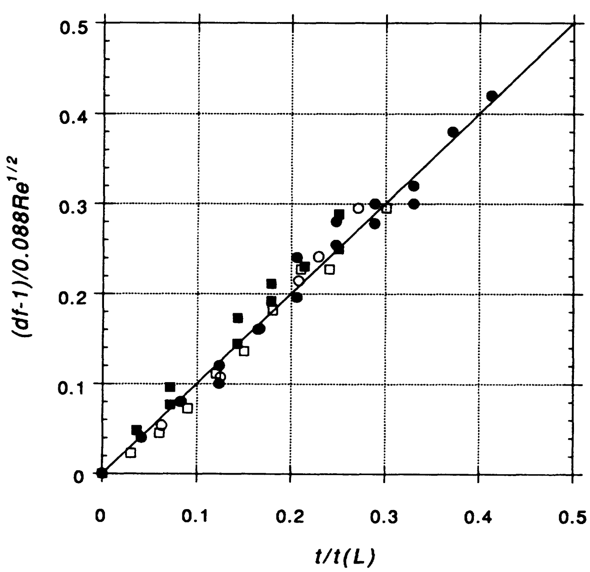

FIG. 5. $[d_f(t) - 1]/0.088\mathrm{Re}^{1/2}$ versus $t/t(L)$. ■, Re = 35; •, Re = 33; □, Re = 25; ∘, Re = 18. A perfect fit would locate all the points on the first bissectrice.

on a compact support and the fractal dimension of the line $d_f(t_M)$ is about 3; from (13), an estimation of t_M is

$$\frac{t_M}{t(L)} = \frac{22.7}{\mathrm{Re}^{1/2}}. \tag{14}$$

In usual laboratory conditions, the micromixing time t_M, qualitatively measured, for example, by the sudden appearance of the product of a chemical reaction (so-called "mixing transition" [16]) is of the order or about a fraction of the turnover time of the large scales. According to (14), we have $t_M/t(L) = 0.7$ for Re = 10^3.

Since the three-dimensional compactness of the small scales is usually achieved within time intervals of the order of $t(L)$, these results might not be affected by the presence of a confinement altering the large-scale structure of the flow and may also hold in finite-size systems [17].

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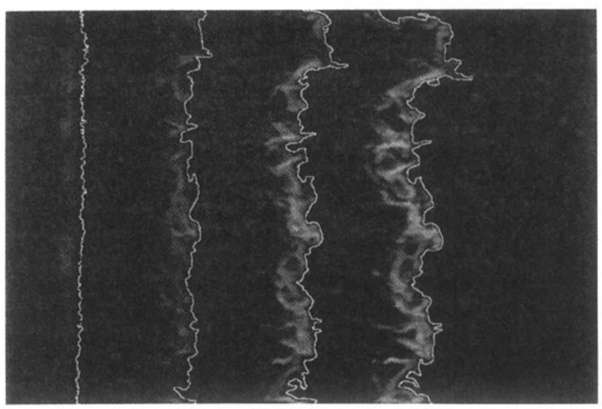

FIG. 1. Evolution, at different consecutive instants spaced by steps of 2/25 sec, of an advected tracer ribbon in a grid-generated turbulence. Re = 33. The external frontier of the ribbon is particularized by the white contour. The integral length scale L is 2.2 cm and the height of the picture covers $30L$.