Lévy Flights in Random Environments

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(Received 2 February 1994)

We consider Lévy flights characterized by the step index f in a quenched isotropic short-range random force field to one loop order. By means of a dynamic renormalization group analysis, we find that the dynamic exponent z for $f < 2$ locks onto f, independent of dimension and *independent* of the presence of weak quenched disorder. The critical dimension for $f < 2$ is given by $d_c = 2f - 2$. For $d < d_c$ the disorder is *relevant*, corresponding to a nontrivial fixed point for the force correlation function. We also discuss the behavior of the subleading diffusive term.

PACS numbers: 05.40.+j, 02.50.–r, 44.30.+v, 47.55.–t

There is current interest in the dynamics of fluctuating manifolds in quenched random environments [1]. The simplest case is that of a random walker in a random environment, corresponding to a zero-dimensional fluctuating manifold. This problem has been treated extensively in the literature and many results are known [2].

In the case of ordinary Brownian motion in a pure environment without disorder, the statistics of the walk is given by a Gaussian distribution with a mean square deviation proportional to the number of steps or the elapsed time,

$$\langle r^2(t)\rangle \propto t^{2/z}, \tag{1}$$

where the dynamic exponent $z = 2$.

There are, however, many interesting processes in nature that are characterized by anomalous diffusion with $z \neq 2$ due to the statistical properties of the environments [2]. Examples are found in chaotic systems [3], which generally lead to superdiffusion with $z < 2$; subdiffusion with $z > 2$ is encountered in constrained systems like fractals [4].

Independent of the spatial dimension d, Brownian motion traces out a manifold of fractal dimension $d_F = 2$ [5]. In the presence of a quenched disordered force field in d dimensions, the Brownian walk is unaffected for $d > d_F$; i.e., for d larger than the critical dimension $d_c = d_F$ the walk is transparent and the dynamic exponent $z = 2$. Below the critical dimension $d_c = 2$ the long time character of the walk is changed to subdiffusive behavior with $z > 2$ [6,7]. In $d = 1$, $\langle r^2(t)\rangle \propto [\ln t]^4$, independent of the strength of the quenched disorder [8].

Lévy flights constitute an interesting generalization of ordinary Brownian walks. Here the step size is drawn from a Lévy distribution characterized by the step index f [5]. The distribution has a long-range algebraic tail corresponding to large but infrequent steps, so-called *rare events*. The "built in" superdiffusive characteristics of Lévy flights have been used to model a variety of physical processes such as self-diffusion in micelle systems, and transport in heterogeneous rocks [9].

In the present Letter we consider Lévy flights in the presence of a quenched random force field and examine the interplay between the built in superdiffusive behavior of the Lévy flights and the pinning effect of the random environment, generally leading to subdiffusive behavior. Details of our analysis will be published elsewhere.

We discuss Lévy flights in terms of a Langevin equation with "power law" noise. In an arbitrary drift force field $\mathbf{F}(\mathbf{r})$, representing the quenched disordered environment,

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{F}(\mathbf{r}(t)) + \boldsymbol{\eta}(t). \tag{2}$$

Here $\boldsymbol{\eta}$ is the instantly correlated power law white noise with the isotropic distribution

$$p(\boldsymbol{\eta})d^d\eta \propto \eta^{-1-f}d\eta, \tag{3}$$

characterized by the step index f [10]. For normalizability we have introduced a lower cutoff $\eta \sim a$ of the order of a microscopic length a and chosen $f > 0$. For $f > 2$ the second moment, $\langle \eta^2\rangle = \int p(\boldsymbol{\eta})\eta^2 d^d\eta$, is finite and a characteristic step size is given by $\sqrt{\langle \eta^2\rangle}$. For $1 < f < 2$ the second moment diverges, but the mean step, $\langle \eta\rangle$, is finite. In the interval $0 < f < 1$ the first moment diverges and even a mean step size is not defined.

The noise $\boldsymbol{\eta}$, describing the consecutive Lévy steps, drives the position $\mathbf{r}$ of the walker. For the quenched force field $\mathbf{F}(\mathbf{r})$ we assume a Gaussian distribution, $p(\mathbf{F}) \propto \exp[-\frac{1}{2}\int d^d r d^d r' F^\alpha(\mathbf{r})\Delta^{\alpha\beta}(\mathbf{r},\mathbf{r}')^{-1}F^\beta(\mathbf{r}')]$ with force correlation function

$$\langle F^\alpha(\mathbf{r})F^\beta(\mathbf{r}')\rangle_F = \Delta^{\alpha\beta}(\mathbf{r}-\mathbf{r}'). \tag{4}$$

For $\mathbf{F}(\mathbf{r}) = \mathbf{0}$ we obtain from $P(\mathbf{r},t) = \langle \delta(\mathbf{r}-\mathbf{r}(t))\rangle$ the solution of Eq. (2), and averaging according to Eq. (3), the scaling form [5],

$$P(\mathbf{r},t) = \int \frac{d^d k}{(2\pi)^d}\exp(i\mathbf{k}\cdot\mathbf{r} - D_1 k^\mu |t|) = |t|^{-\frac{d}{\mu}} G\left(r/|t|^{\frac{1}{\mu}}\right). \tag{5}$$

D_1 is a diffusion coefficient setting the time scale.

The scaling exponent μ depends on f. For $f > 2$, $\mu = 2$ and the scaling function $G(x) = \exp(-x^2)$; this is a consequence of the central limit theorem. For $f < 2$, $\mu = f$. In the scaling regime [11] we deduce from Eq. (5),

$$\langle r^2(t)\rangle \propto \int P(\mathbf{r},t)\, r^2 d^d r \propto t^{\frac{2}{\mu}} \tag{6}$$

and from Eq. (1) $z = \mu$. We note that the fractal dimension of a Lévy flight is $d_F = \mu$ [5].

In the presence of the quenched force field we recast the problem given by Eq. (2) in terms of the Fokker-Planck equation following from Eq. (5):

$$\frac{\partial P(\mathbf{r},t)}{\partial t} = -\nabla[\mathbf{F}(\mathbf{r})P(\mathbf{r},t)] + D_1\nabla^\mu P(\mathbf{r},t) + D_2\nabla^2 P(\mathbf{r},t), \tag{7}$$

including the ordinary diffusion term, $D_2\nabla^2 P$, originating from the low $\boldsymbol{\eta}$ part of the distribution $p(\boldsymbol{\eta})$. The "fractional" gradient operator ∇^μ is the Fourier transform of $-k^\mu$ and is a spatially nonlocal integral operator reflecting the long-range steps.

There are a variety of techniques available in order to treat the random Fokker-Planck equation (7). Applying the Martin-Siggia-Rose formalism in functional form [12] and using either the replica method [2] or an explicit casual time dependence [6], one can average over the quenched force field and construct an effective field theory. A more direct method amounts to an expansion of the Fokker-Planck equation (7) in powers of the force field and an average over products of $\mathbf{F}(\mathbf{r})$ according to Eqs. (4) [13].

Defining $P(\mathbf{k},\omega) = \int d^d r\, dt \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})P \times (\mathbf{r},t)\theta(t)$, where $\theta(t)$ is the step function, we obtain, introducing the dimensionless coupling strength λ for the vertex,

$$(-i\omega + D_1 k^\mu + D_2 k^2)P(\mathbf{k},\omega) = P_0(\mathbf{k}) - i\lambda\mathbf{k}\int\frac{d^d p}{(2\pi)^d}\mathbf{F}(\mathbf{k}-\mathbf{p})P(\mathbf{p},\omega). \tag{8}$$

The force field is averaged according to Eq. (4), i.e., $\langle F^\alpha(\mathbf{k})F^\beta(\mathbf{p})\rangle_F = \Delta^{\alpha\beta}(\mathbf{k})\,(2\pi)^d\,\delta(\mathbf{k}+\mathbf{p})$ for all pairwise force contractions; $P_0(\mathbf{k}) = P(\mathbf{k}, t=0)$ is the initial distribution. We consider the case of isotropic zero-range force correlations, i.e., $\Delta^{\alpha\beta}(\mathbf{k}) = \Delta\delta^{\alpha\beta}$; the general case has been discussed in Refs. [2,13,15]. We introduce a microscopic UV cutoff and assume $0 < k, p < 1$. Iterating Eq. (8) and identifying self-energy, vertex, and force correlation corrections to first order in Δ, we find divergent contributions to the subleading term $D_2 k^2$ for $d < \mu$ and to the vertex λ for $d < 2\mu - 2$. In order to disentangle the breakdown of primitive perturbation theory and deduce the scaling properties of the force averaged distribution $\langle P(\mathbf{r},t)\rangle_F$ and the mean square displacement $\langle\langle r^2(t)\rangle\rangle_F$, we carry out a renormalization group analysis, following the momentum shell integration method [13,14]. Averaging over the force in the shell $e^{-l} < k, p < 1$, we thus obtain the "corrected" Fokker-Planck equation:

$$[-i\omega + D_1 k^\mu + (D_2 + \delta D_2)k^2]P(\mathbf{k},\omega) = P_0(\mathbf{k}) - i(\lambda + \delta\lambda)\mathbf{k}\int\frac{d^d p}{(2\pi)^d}\mathbf{F}(\mathbf{k}-\mathbf{p})P(\mathbf{p},\omega), \tag{9}$$

and the force correlation function

$$\langle F^\alpha(\mathbf{k})F^\beta(\mathbf{p})\rangle_F = (\Delta + \delta\Delta)\delta^{\alpha\beta}(2\pi)^d\delta(\mathbf{k}+\mathbf{p}) \tag{10}$$

for $0 < k, p < e^{-l}$. Note that there is no correction to the leading Lévy term. For small values of the scale parameter l the corrections δD_2, $\delta\lambda$, and $\delta\Delta$ are proportional to l. From the diagrammatic contributions given in Refs. [6,13], evaluated in the Lévy case to one loop order, we obtain

$$\delta D_2 = A'\frac{D_1(d-\mu) + D_2(d-2)}{(D_1 + D_2)^2}\lambda^2\Delta l, \tag{11}$$

$$\delta\lambda = -C'\frac{\lambda^3\Delta}{(D_1 + D_2)^2}l, \tag{12}$$

$$\delta\Delta = -B'\frac{\lambda^2\Delta^2}{(D_1 + D_2)^2}l, \tag{13}$$

where A', B', and C' are geometric factors. In order to derive the "renormalized" Fokker-Planck equation, we introduce scaled quantities $\mathbf{k}' = \mathbf{k}e^l$, $\mathbf{p}' = \mathbf{p}e^l$, $\omega' = \omega e^{\alpha(l)}$, $P'(\mathbf{k}',\omega') = P(\mathbf{k},\omega)e^{-\alpha(l)}$, and $\mathbf{F}'(\mathbf{k}') = \mathbf{F}(\mathbf{k})e^{-\beta(l)}$ such that $0 < k', p' < 1$.

From the renormalized Fokker-Planck equation and force correlation function, adjusting $\beta(l)$ so that $\lambda = 1$, setting $\alpha(l) = \int_0^l z(l')dl'$, choosing $z(l) = \mu$ in order to fix $D_1(l) = D_1$, we read off the renormalization group equations for D_2 and Δ,

$$\frac{dD_2}{dl} = (\mu - 2)D_2 + A\frac{D_1(d-\mu) + D_2(d-2)}{(D_1 + D_2)^2}\Delta, \tag{14}$$

$$\frac{d\Delta}{dl} = (2\mu - d - 2)\Delta - B\frac{\Delta^2}{(D_1 + D_2)^2}. \tag{15}$$

Here $A = (1/2d)S_d/(2\pi)^d$, $B = (3/d - 1)S_d/(2\pi)^d$, and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface area of a d-dimensional sphere.

Proceeding with the discussion of Eqs. (14) and (15), we note that for $\mu < 1 + d/2$ Eqs. (14) and (15) have the trivial fixed points $D_2^* = 0$ and $\Delta^* = 0$, indicating that

(i) the subleading term, D_2k^2, scales to zero compared with the leading Lévy term and (ii) the quenched disorder, characterized by Δ, is *irrelevant*. The long-range Lévy steps predominate and control the scaling behavior. In a plot of $D_2(l)$ versus $\Delta(l)$ the size of the linear scaling regime, i.e., the region where the trajectories flow to the fixed point with constant slope, depends on μ and becomes largest for $\mu = d$, precisely the case where perturbation theory begins to yield a divergent contribution to D_2. For $1 + d/2 < \mu < 2$ nontrivial fixed points emerge for D_2 and Δ,

$$D_2^* = (A/B)\frac{D_1(d-\mu)(2\mu-d-2)}{2-\mu+(A/B)(2\mu-d-2)(2-d)}, \tag{16}$$

$$\Delta^* = (1/B)(2\mu - d - 2)(D_1 + D_2^*)^2. \tag{17}$$

The fixed point D_2^* indicates that the subleading diffusive term D_2k^2 now yields a contribution compared to the Lévy term D_1k^μ. The fixed point Δ^* shows that for d less than the critical dimension $d_c = 2\mu - 2$ the quenched disorder becomes *relevant*. The fixed point value of the diffusion coefficient, D_2^*, is negative since the pinning environment tends to reduce the ordinary diffusion from $D_2^* = 0$ for $\mu < 1 + d/2$. We also note that unlike the case of Brownian motion the critical dimension $d_c = 2\mu - 2$ is less than the fractal dimension $d_F = \mu$. For $\mu \to 2$ it follows from Eq. (16) that $D_2^* \to -D_1$ of that the Lévy term D_1k^μ precisely cancels with the diffusive term D_2k^2 in the Fokker-Planck equation (8); this is consistent with the fact that there is no correction to first loop order or more precisely to first order in $d_c \to d$ in the Brownian case [6].

In order to derive the scaling properties of the force averaged distribution $\langle P(\mathbf{k},\omega)\rangle_F$ and the mean square displacement $\langle\langle r^2(t)\rangle\rangle_F$, we use the methods discussed in Refs. [13,14]. From the derivation of the renormalization group equations we infer the scaling relation $\langle P(\mathbf{k},\omega,\Delta)\rangle_F = e^{\alpha(l)}\langle P(\mathbf{k}e^l, \omega e^{\alpha(l)}, \Delta(l))\rangle_F$. In the vicinity of either fixed point we have, setting $\alpha(l) \propto \mu l$, $\langle P(\mathbf{k},\omega,\Delta)\rangle_F = e^{\mu l}\langle P(\mathbf{k}e^l, \omega e^{\mu l}, \Delta^*)\rangle_F$. Choosing $ke^l \sim 1$ we obtain the scaling form

$$\langle P(\mathbf{k},\omega,\Delta)\rangle_F = k^{-\mu}L(k/\omega^{1/\mu}), \tag{18}$$

where L is a scaling function. From Eq. (18) follows directly (see [11])

$$\langle\langle r^2(t)\rangle\rangle_F \propto t^{2/\mu} = t^{2/z}. \tag{19}$$

For Lévy flights in random quenched environment we have shown that the dynamic exponent z locks onto the scaling index μ, depending on the Lévy step index f, *independent* of the presence of weak quenched disorder. The long-range superdiffusive behavior characteristic of Lévy flights enables the walker to escape the inhomogeneous pinning environment, and the long time behavior is the same as in the pure case. We have also identified a critical dimension $d_c = 2\mu - 2$, depending on the scaling exponent μ. For $d < d_c$ the weak disorder becomes *relevant* as shown by the emergence of a nontrivial fixed point.

Bouchaud *et al.* [16] have given a heuristic argument yielding the critical dimension $d_c = \mu$ in the Lévy case. This result is at variance with the critical dimension $d_c = 2\mu - 2$ given here based on a renormalization group analysis. Note, however, that perturbation theory yields a divergent contribution to the subleading diffusive term for $d = \mu$. We have not entirely appreciated the discrepancy between Bouchaud's argument and the present analysis. It clearly would be of interest to construe a qualitative heuristic argument for the critical dimension $d_c = 2\mu - 2$ given here and the insensitivity of the dynamic exponent $z = \mu$ to the weak quenched disorder.

The present analysis opens up several avenues to pursue such as the interplay between the temporal and spatial features of Lévy flights in quenched environments and the role played by the range and vector nature of the random force field. These problems will be dealt with in a future publication.

The author wishes to thank L. Mikheev, A. Svane, and K. Bækgaard Lauritsen for helpful discussions. This work was supported by the Danish Research Council, Grant No. 11-9001.

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[10] The Langevin equation (2) is the continuum limit of the corresponding difference equation defined for a discrete time step Δ. For $f > 1$ this limit is attained by letting the lower cutoff for $p(\eta)$ approach infinity; for $f < 1$ the

cutoff must go to zero. Note that these "renormalizations" are not observable; the problem at hand is defined by the Langevin equation (2) or, correspondingly, the Fokker-Planck equation (7).

[11] For Lévy flights the scaling regime must be defined with some care. Since for $f < 2$, $\langle \eta^2 \rangle$ diverges, it clearly follows that $\langle r^2(t) \rangle = \langle \sum_i \eta_i^2 \rangle$ diverges as well. However, considering Lévy flights within a volume of linear extent L, it follows from Eq. (5) that for (i) $L \gg t^{1/\mu}$, $\langle r^2 \rangle_L \propto tL^{2-\mu}$, i.e., diverging for $L \rightarrow \infty$, and for (ii) $L \ll t^{1/\mu}$, $\langle r^2 \rangle_L \propto t^{2/\mu}$. The latter regime is the scaling regime in the present discussion. Further discussion of these points is deferred to a future publication.

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