Analytical Solution to the Quantum Field Theory of Self-Phase Modulation with a Finite Response Time

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(Received 20 December 1993)

The quantum theory of a field in 1+1 dimensions coupled to localized oscillators is developed. The solution to the Heisenberg equation for the field is given in closed form. It is shown that the nonlinearity of the medium is inevitably accompanied by phase noise of the field. This noise explains the preferential growth of the Stokes wave for short propagation distances.

PACS numbers: 42.50.Lc, 42.65.Dr

In recent years the study of quantum field theories in spaces with reduced dimensions has been the focus of considerable attention [1]. This interest is justified in part by the integrability of some of the models and the physical insight that exact solutions can bring to theories in 3+1 dimensions. In certain cases, however, the reduced dimensionality corresponds to physical systems like strings, fibers, membranes, or surfaces where one or two dimensions of space are either absent or do not play a significant role. In this Letter we present an analytically solvable theory for a field in 1+1 dimensions coupled to localized oscillators. This theory provides a simple and self-consistent model for the quantum propagation of pulses in single mode and dispersionless optical fibers. Our analysis reveals that the oscillators, which represent molecular vibrations, are not only responsible for a delayed nonlinearity of the field, but also produce phase noise. The exact relation between the response function of the nonlinearity and the spectrum of the noise follows from the canonical structure of the theory.

An important goal of this work is to understand the quantum mechanical evolution of the electromagnetic field fluctuations in nonlinear optical media. Our construction provides nonperturbative expressions for all the moments of the field in the limit where the medium is dispersionless and one transverse mode can be studied independently from the others. It is then a natural starting point for a perturbative analysis of other effects. On a more fundamental level, our model suggests that medium induced nonlinearities in general, unlike nonlinearities present at the elementary level, are accompanied by noise. Our analysis shows how the preservation of commutation relations can be used to probe the physical properties of this noise.

Let $\hat{a}(\beta)$ and $\hat{a}^\dagger(\beta)$ be the annihilation and creation operators for an excitation of the field with wave number β and frequency $\omega(\beta)$. If only modes within a narrow range of wave numbers centered around β_0 are excited, one can assume a linear dispersion relation, $\omega(\beta) = \omega_0 + v_g(\beta - \beta_0)$. Defining the slowly varying envelope $\hat{A}(z,t)$ by

$$\hat{A}(z,t) = e^{i\omega_0 t - i\beta_0 z}\sqrt{\frac{v_g}{2\pi}}\int_{-\infty}^{\infty} d\beta\, e^{i\beta z}\hat{a}(\beta,t), \qquad (1)$$

we find, from $\hat{H}_0 = \hbar\int d\beta\, \omega(\beta)\hat{a}^\dagger(\beta)\hat{a}(\beta)$, the Hamiltonian,

$$\hat{H}_0 = \hbar\int\frac{dz}{v_g}\left\{\omega_0\hat{A}^\dagger\hat{A} + \frac{iv_g}{2}(\partial_z\hat{A}^\dagger\hat{A} - \hat{A}^\dagger\partial_z\hat{A})\right\}. \quad (2)$$

The envelope was normalized to satisfy the equal time commutation relations $[\hat{A}(z,t),\hat{A}(z',t)] = 0$ and $[\hat{A}(z,t),\hat{A}^\dagger(z',t)] = v_g\delta(z-z')$. In the absence of nonlinearity each annihilation operator evolves independently according to $\hat{a}(\beta,t) = \exp[-i\omega(\beta)t]\hat{a}(\beta,0)$. In this case, the space time evolution of $\hat{A}$ occurs only due to the bandwidth of the superposition in (1). The nonlinear coupling of the field to the medium is introduced through the interaction Hamiltonian,

$$\hat{H}_{\rm int} = -\hbar\kappa\int dz\, \hat{n}_{\rm NL}\hat{A}^\dagger\hat{A}. \qquad (3)$$

The Hermitian operator $\hat{n}_{\rm NL}(z,t)$ commutes at equal times with the envelope and represents the matter degrees of freedom affecting the field. Comparing (2) and (3), $\hat{n}_{\rm NL}(z,t)$ can be interpreted as a local change of the carrier frequency, or, with $\kappa = -v_g^{-1}d\omega_0/dn$, as a small change of the local index of refraction. Using the coordinates $(z, \tau = t - z/v_g)$, the Heisenberg equation for the envelope is [2]

$$\frac{\partial}{\partial z}\hat{A}(z,\tau) = i\kappa\hat{n}_{\rm NL}(z,\tau)\hat{A}(z,\tau). \qquad (4)$$

We emphasize that canonical commutators apply at equal t and not at equal τ.

The Hermiticity of $\hat{n}_{\rm NL}$ has an important consequence. Consider the operator $\hat{I}(z,\tau) = \hat{A}^\dagger(z,\tau)\hat{A}(z,\tau)$ measuring the flux of quanta (photons) at z. Using (4) and its Hermitian conjugate, we find that $\hat{I}$ is z independent. This indicates that even though the phase of the envelope is affected by the medium, the flux of photons travels unchanged at the group velocity. If all photons were of the

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same frequency, $\hat{I}$ would be proportional to the z component of the Poynting vector. However, $\hat{A}(z,\tau)$ contains modes of a finite bandwidth coupled with the medium. Since photons can exchange energy with the medium and simultaneously change frequency, the conservation of the flux of photons does not imply the conservation of the energy of the pulses.

It is convenient at this point to cast our theory in terms of input and output operators. The pulses are incident from a linear region ($z < 0$) onto the medium, and coupled out again to another linear region ($z > l$). To simplify we will assume that the three media have well matched transverse modes so that no reflection occurs at the boundaries. We denote by "i" and "o" the fields at $z = 0$ and $z = l$, respectively. In the linear sections, the fields are functions of τ only and one then has

$$[\hat{A}_i(\tau), \hat{A}_i(\tau')] = 0; \quad [\hat{A}_i(\tau), \hat{A}_i^\dagger(\tau')] = \delta(\tau - \tau'), \tag{5}$$

$$[\hat{A}_o(\tau), \hat{A}_o(\tau')] = 0; \quad [\hat{A}_o(\tau), \hat{A}_o^\dagger(\tau')] = \delta(\tau - \tau'). \tag{6}$$

Indeed one can express the i fields in terms of the fields in the linear input section at some large and negative time $t_0 \to -\infty$. For example, $\hat{A}_i(\tau) = \hat{A}(v_g(t_0 - \tau), t_0)$, where in this last field we reverted to the original coordinates $(z,t) = (z, \tau + z/v_g)$. The equal time commutators for $\hat{A}$ and $\hat{A}^\dagger$ then imply (5). The same reasoning can be applied to the o fields with $t_0 \to \infty$. When the boundaries are partially transmitting, the zero point fluctuations from the backward traveling channels ensure that the input and output commutators are satisfied. Writing $\hat{I}(\tau) = \hat{A}_i^\dagger(\tau)\hat{A}_i(\tau)$, we clearly have from (5)

$$[\hat{I}(\tau), \hat{I}(\tau')] = 0. \tag{7}$$

The final part of our theory specifies the time evolution of $\hat{n}_{\mathrm{NL}}$. Fortunately it is not necessary to know the detailed microscopic Hamiltonian of the material to determine its optical properties. The interaction Hamiltonian indicates that the nonlinear index couples to the envelope only through its intensity. Optical nonlinearities being usually very small, it is sufficient to expand $\hat{n}_{\mathrm{NL}}$ in (4) up to first order in its dependence on $\hat{I}$:

$$\kappa \hat{n}_{\mathrm{NL}}(z,\tau) = \int_{-\infty}^{\infty} d\tau'\, f(\tau')\hat{I}(\tau - \tau') + \hat{m}(z,\tau). \tag{8}$$

Returning to the (z,t) coordinates and writing $\hat{I}(\tau - \tau') = \hat{A}^\dagger(z, t-\tau')\hat{A}(z, t-\tau')$ in the integral, $f(\tau)$ is seen to describe a delayed Kerr-type nonlinearity and vanishes by causality when $\tau < 0$. Owing to the extremely short response time of the Kerr effect in material like silica (~ 5 fs), one usually considers this nonlinearity instantaneous when studying the classical propagation of long pulses. Several authors [3,4] pointed out, however, that quantum theories of light in instantaneous Kerr media [5] are ill defined in the absence of dispersion due to the infinite bandwidth of the vacuum fluctuations coupling to any frequency window of interest. Hence, even though one is interested in the quantum evolution of long pulses, reference to the much shorter response time of the nonlinearity is unavoidable.

The operator $\hat{m}(z,\tau)$ in (8) describes the quantum and thermal fluctuations present in $\hat{n}_{\mathrm{NL}}$ in the absence of optical field. The photon flux being unaffected by its propagation in the waveguide, one must have

$$[\hat{I}(\tau), \hat{m}(z,\tau')] = 0. \tag{9}$$

The nonlinear index is then described by a traveling distortion on an independent noise background. Introducing (8) into (4) produces a nonlinear, noninstantaneous, and stochastic Heisenberg equation describing the evolution of the envelope field. We note that a response function was introduced already in a model for the Kerr effect by Blow *et al.* [3]. Although the need for the attendant noise source was anticipated by these authors, they did not indicate where it should be inserted.

We model the noise background as a collection of localized and independent harmonic oscillators:

$$\hat{m}(z,\tau) = \int_0^\infty d\omega \frac{\sqrt{W(\omega)}}{2\pi} \{\hat{d}_\omega^\dagger(z) e^{i\omega\tau} + \text{H.c.}\}, \tag{10}$$

where the spectral weighting function $W(\omega)$ is, as yet, unspecified. The operators $\hat{d}_\omega(z)$ and $\hat{d}_\omega^\dagger(z)$, which are independent from the envelope, obey the commutation relation $[\hat{d}_\omega(z), \hat{d}_{\omega'}^\dagger(z')] = \delta(\omega - \omega')\delta(z - z')$. All other commutators involving these operators vanish. Note that the material oscillators at different locations are completely decoupled. This is consistent with a picture where $\hat{n}_{\mathrm{NL}}$ is created by localized molecular vibrations. Such oscillators are responsible for Raman scattering. Acoustical vibrations, which couple different parts of the waveguide, contribute also to the noise in the electromagnetic field. We do not include them in our theory as they do not lead to closed form solutions for $\hat{A}(z,\tau)$. Using (7) to (10), the nonlinear index operators at different locations are seen to commute at different times and, in particular, at equal τ. This allows (4) to be integrated as if $\hat{n}_{\mathrm{NL}}$ were a c number, leading to the closed form solution $\hat{A}(z,\tau) = \exp[i\kappa \int_0^z dz'\, \hat{n}_{\mathrm{NL}}(z',\tau)]\hat{A}_i(\tau)$. Using again (9), we can write the following input-output connection:

$$\hat{A}_o(\tau) = \exp[i\hat{\theta}(\tau)] \exp[i\hat{\phi}(\tau)] \hat{A}_i(\tau), \tag{11}$$

where $\hat{\phi}(\tau) = l \int_0^\infty d\tau'\, f(\tau')\hat{I}(\tau - \tau')$ is the self-phase modulation operator and $\hat{\theta}(\tau) = \int_0^l dz\, \hat{m}(z,\tau)$ is the phase noise added by the medium. The relative order of $\hat{A}_i$ and $\exp(i\hat{\phi})$ is important but the noise exponential, which commutes with the field factors, can be inserted anywhere. Using this solution and the input commutators we find, after some algebra (Appendix),

$$[\hat{A}_o(\tau), \hat{A}_o(\tau')] = \hat{O}_1 \hat{A}_o(\tau)\hat{A}_o(\tau'), \quad (12)$$

$$[\hat{A}_o(\tau), \hat{A}_o^\dagger(\tau')] = \delta(\tau - \tau') + \hat{O}_2 \hat{A}_o^\dagger(\tau')\hat{A}_o(\tau), \quad (13)$$

$$\hat{O}_1 = (1 - e^{-il\alpha}) + e^{-il\alpha}[e^{i\hat{\theta}(\tau)}, e^{i\hat{\theta}(\tau')}]e^{-i\hat{\theta}(\tau')}e^{-i\hat{\theta}(\tau)},$$

$$\hat{O}_2 = (e^{-il\alpha} - 1) + e^{-il\alpha}[e^{i\hat{\theta}(\tau)}, e^{-i\hat{\theta}(\tau')}]e^{-i\hat{\theta}(\tau)}e^{i\hat{\theta}(\tau')},$$

where $\alpha = f(\tau' - \tau) - f(\tau - \tau')$. These commutators satisfy (6) if $\hat{O}_1 = \hat{O}_2 = 0$. One verifies from the identity $[e^A, e^B] = e^B e^A(e^{[A,B]} - 1)$, which holds if $[A, B]$ is a c number, that this happens when $[\hat{\theta}(\tau), \hat{\theta}(\tau')] = il\alpha$. Using (10) to compute this last commutator, we find that the spectral weighting function must be related to the imaginary part of the Fourier transform of the response function by

$$W(\omega) = 4\pi\tilde{f}''(\omega) \geq 0. \quad (14)$$

Here $\tilde{f}''(\omega) = \int d\tau \sin(\omega\tau) f(\tau)$ vanishes only if $f(\tau)$ is even. Since $f(\tau < 0) = 0$, the only possible symmetric response function is proportional to a delta function. The resulting instantaneous nonlinearity leads, as mentioned already, to a singular quantum field theory. Hence, noise sources are always required to preserve the canonical structure of the theory. The Hermiticity of $\hat{m}$ requires the positivity of $\tilde{f}''(\omega > 0)$. Writing $\hat{n}_{\rm NL}$ as a weighted sum of harmonic oscillators, this condition follows from the positivity of the damping coefficients. The inequality in (14) is then a stability condition verified experimentally for the optical field by the positivity of the Raman gain on the Stokes side of the carrier frequency only [6]. Below, the material oscillators will be assumed in thermal equilibrium:

$$\langle \hat{d}_\omega^\dagger(z)\hat{d}_{\omega'}(z')\rangle = \delta(\omega - \omega')\delta(z - z')n_{\rm th}(\omega), \quad (15)$$

where $n_{\rm th}(\omega) = [\exp(\hbar\omega/kT) - 1]^{-1}$ is the Bose-Einstein distribution. The phase noise operator then has a Gaussian statistics.

Our theory differs from that of Blow *et al.* [3] by the presence of the noise exponential in (11). This factor, which makes the theory self-consistent, has physically observable consequences as we now show. Consider a coherent input state $|\psi\rangle$ corresponding to a monochromatic pump at the carrier frequency. One then has $\hat{A}_i(\tau)|\psi\rangle = \sqrt{I_p}|\psi\rangle$. From (11), the field autocorrelation function for this state has the form $\langle \hat{A}_o^\dagger(0)\hat{A}_o(\tau)\rangle = I_p\langle e^{-i\hat{\theta}(0)}e^{i\hat{\theta}(\tau)}\rangle\langle e^{-i\hat{\phi}(0)}e^{i\hat{\phi}(\tau)}\rangle$. The two expectation values can be evaluated for any response function. Recall from (7) that $e^{-i\hat{\phi}(0)}e^{i\hat{\phi}(\tau)} = e^{-i\hat{\phi}(0)+i\hat{\phi}(\tau)}$. By using the normal ordering formula (A1) with $g(s) = il\{f(s) - f(s - \tau)\}$, we find

$$\langle e^{-i\hat{\phi}(0)}e^{i\hat{\phi}(\tau)}\rangle = \exp\left(I_p \int ds\, \{e^{g(s)} - 1\}\right). \quad (16)$$

On the other hand, $\langle e^{-i\hat{\theta}(0)}e^{i\hat{\theta}(\tau)}\rangle = e^{i\frac{1}{2}\alpha}\langle e^{-i\hat{\theta}(0)+i\hat{\theta}(\tau)}\rangle$, where this time $\alpha = f(\tau) - f(-\tau)$. The expectation value of the last exponential is computed from (15) using the identity $\langle \exp(X)\rangle = \exp(\frac{1}{2}\langle X^2\rangle)$ if X has a Gaussian statistics with a vanishing average. We finally find the following contribution to the autocorrelation function from the noise source:

$$\langle e^{-i\hat{\theta}(0)}e^{i\hat{\theta}(\tau)}\rangle = e^{-i\frac{1}{2}\alpha}e^{-lJ(\tau)}, \quad (17)$$

where $J(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\tilde{f}''(\omega)\sin^2(\omega\tau/2)\coth(\hbar\omega/2kT)$. The autocorrelation function can be Fourier transformed to give the spectrum of the radiation leaving the medium. Figure 1 displays output spectra of a medium with a single resonance Ω and with a damping coefficient Γ [7]. The resulting response function is

$$f(\tau) = \frac{K\Omega^2}{\sqrt{\Omega^2 - \Gamma^2/4}}u(\tau)e^{-\frac{\Gamma}{2}\tau}\sin(\sqrt{\Omega^2 - \Gamma^2/4}\,\tau), \quad (18)$$

where $u(\tau)$ is the step function and $K = \int d\tau\, f(\tau)$. For short propagation distances, one can identify in these plots the contributions from individual low order Feynman diagrams (Fig. 2). Expanding (16) and (17) to lowest order in l, we find that the noise expectation value grows linearly with l whereas self-phase modulation contributes only to second order. The short distance output spectrum is then dominated by the new noise source introduced in our theory. The corresponding processes are shown in Fig. 2(a) (Stokes) and Fig. 2(b) (anti-Stokes). At low temperatures such that $kT < \hbar\Omega$, the process of

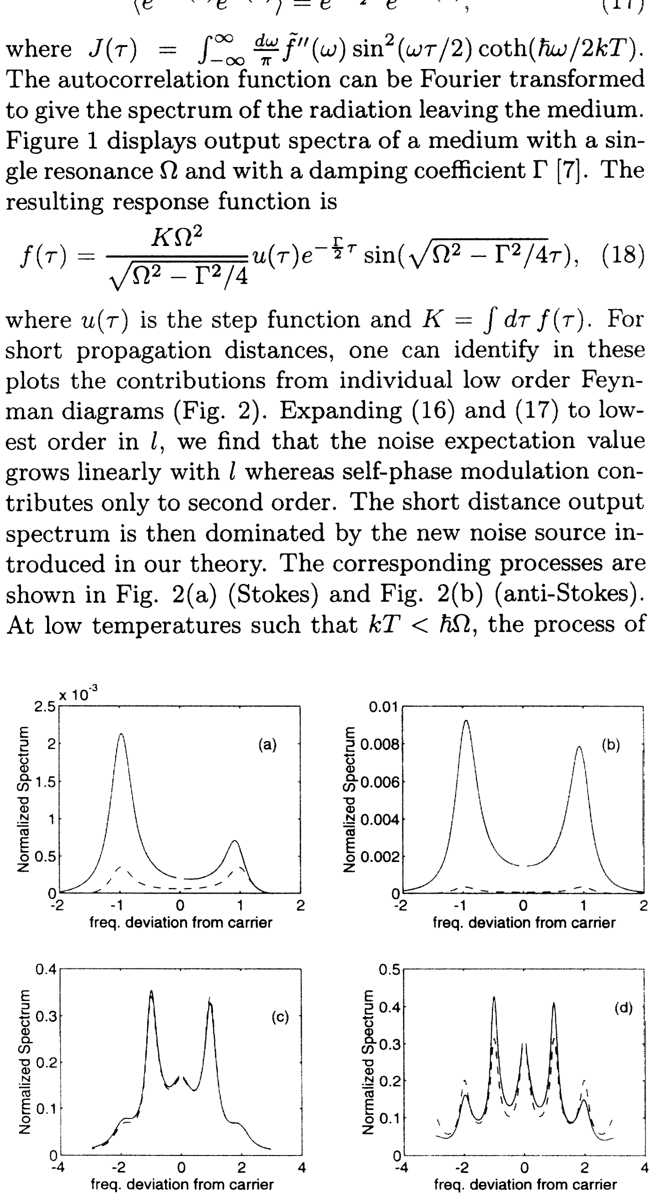

FIG. 1. Normalized power spectra as functions of the frequency deviation from the pump for the medium described by (18). The frequencies are normalized to the resonance. The dashed lines were plotted after removing the noise. All plots use the same pump intensity. Defining $x = K\Omega l/v_g$ and $\gamma = \Gamma/\Omega$ to measure the propagation distance and the width of the resonance, respectively, we have (a) $x = 0.002$, $\gamma = 0.4$, $kT = 0.5\hbar\Omega$; (b) $x = 0.002$, $\gamma = 0.4$, $kT = 5\hbar\Omega$; (c) $x = 0.1$, $\gamma = 0.4$, $kT = 0.5\hbar\Omega$; (d) $x = 0.1$, $\gamma = 0.15$, $kT = 0.5\hbar\Omega$. The spectrum near the pump was removed.

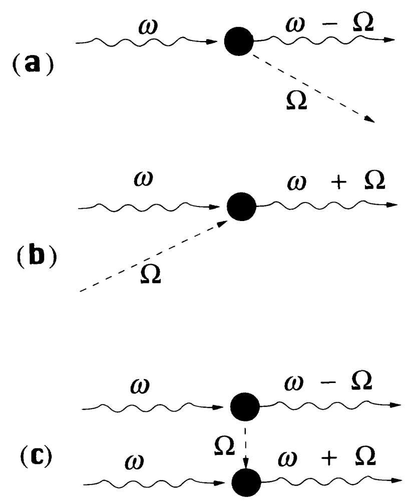

FIG. 2. Photon number preserving Feynman diagrams contributing to spectral broadening for short propagation distances. In these figures, dashed lines represent matter excitations with frequencies close to a resonance of the medium [Ω for a medium described by (18)]. Processes (a) and (b) are absent when $\hat{m}$ is neglected.

Fig. 2(b), in which a photon gains a quantum of energy from the waveguide, is discouraged due to the absence of thermal medium excitations. In this case, most photons scattered from the pump end up with a lower frequency. As Fig. 1(b) shows, partial symmetry of the spectrum is restored at higher temperatures. Note that the spectra computed with the theory in [3] (dashed lines) are symmetric. This is readily understood for short propagation distances since in this case the lowest order Feynman diagram contributing to spectral broadening is shown in Fig. 2(c). Clearly this process, in which the medium participates only as the mediator of the nonlinear wave mixing, produces as many photons on the Stokes and anti-Stokes sides. The short distance output spectrum is then qualitatively different when the noise of the medium is neglected as in [3]. This provides a clear experimental test for our model. The plots 1(c) and 1(d) show spectra at longer propagation distances where the contribution of individual Feynman diagrams can no longer be distinguished. The growth of secondary Stokes and anti-Stokes waves is obvious and is enhanced by a sharper medium resonance. The medium noise is less important in that regime.

In this Letter we presented a model for which a medium-induced nonlinearity requires a Langevin noise source to preserve the canonical structure of the theory. This phase noise is experimentally observable from the preferential growth of Stokes waves at low temperature. We suggest that noise sources are a general feature of nonlinear quantum optics that should be kept in mind when studying the fluctuations of the electromagnetic field. An important aspect of the model developed here is that it provides an example of a nonlinear quantum field theory with an analytic solution. This presents some interest on its own as most of our physical intuition is ultimately based on a few analytic models.

Appendix.—Write (12) as $[\hat{A}\hat{B},\hat{C}\hat{D}] = [\hat{A},\hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B},\hat{D}]$, with $\hat{A} = e^{i\hat{\theta}(\tau)}$, $\hat{B} = e^{i\hat{\phi}(\tau)}\hat{A}_i(\tau)$, $\hat{C} = e^{i\hat{\theta}(\tau')}$, $\hat{D} = e^{i\hat{\phi}(\tau')}\hat{A}_i(\tau')$. In writing this expression we used $[\hat{B},\hat{C}] = [\hat{A},\hat{D}] = 0$. We have, on the other hand $[\hat{B},\hat{D}] = \hat{B}\hat{D}(1 - e^{-il\alpha})$. This last commutator follows from $\hat{A}_i(\tau')e^{i\hat{\phi}(\tau)} = e^{i\hat{\phi}(\tau)}\hat{A}_i(\tau')e^{ilf(\tau-\tau')}$, which is obtained from (5) after normal ordering the exponentials. This can be done using the formula [4]

$$\exp\left(\int ds\, g(s)\hat{I}(\tau - s)\right) =: \exp\left(\int ds\, h(s)\hat{I}(\tau - s)\right) :, \tag{A1}$$

where $h(s) = e^{g(s)} - 1$. Here $: F(\hat{A}_i^\dagger, \hat{A}_i) :$ means that in the Taylor expansion of F, all creation operators are on the left of the annihilation operators. The commutator $[\hat{A},\hat{C}]$, which we try to determine, is left over in (12). The commutator (13) is obtained in a similar way.

This work was sponsored by the Joint Services Electronics Program (DAAL03-92-C-0001) and by the Navy/ONR (N00014-91-3-1579). L.B. acknowledges a 67 NSERC Scholarship. Support by the Alexander von Humboldt Stiftung is gratefully acknowledged by F.X.K.

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