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Nonlocality of a Single Photon Revisited

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A proposal to demonstrate the nonlocality of a single photon is described. This is accomplished without using inequalities, and in a way that brings out a very curious feature of quantum mechanics. Unlike the first proposal to investigate the nonlocality of a single photon due to Tan, Walls, and Collett, this proposal does not require supplementary assumptions, and therefore rules out the whole class of local hidden variable interpretations.

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A few years ago, Tan, Walls, and Collett (TWC) [1] proposed a scheme to demonstrate the nonlocality of a single photon (see also Oliver and Stroud [2]). The idea was that measurements made on the two output channels from a source could violate locality even when only one photon is emitted from the source at a time. The importance of this idea is emphasized by the fact that, as early as 1927 (at the Fifth Solvay Conference), Einstein [3] presented the collapse of a single particle wave packet to a near position eigenstate as a paradigm for nonlocality in quantum mechanics (indeed, one might even say that he anticipated the Einstein-Podolsky-Rosen argument in the context of this example [4]). This was, of course, long before Bell's careful analysis [5] of a two particle system showing that nonlocality is an irremovable feature of quantum mechanics rather than just being a problem with the formalism. Until the work of TWC, it had been implicitly assumed that to get a violation of Bell inequalities one required a two (or more) particle state, and for this reason their idea surprised many people. Unfortunately, the proposal of TWC requires certain supplementary assumptions which quite severely restrict the class of local models it rules out [6,7]. However, that a single photon might exhibit nonlocality is such an exciting idea that it seems worthwhile to look for a demonstration which does not require supplementary assumptions. Here we will consider a source which never emits more than one photon at a time and, on average, emits less than one photon. We will find measurements that allow a demonstration of nonlocality without using inequalities (by analog with the proof in [8]). The experimental configuration we will consider is quite similar to that of TWC, but the analysis is completely different.

First we will see how to prepare states like $e|0\rangle + f|1\rangle$. Figure 1 shows a nonlinear crystal being pumped by a strong laser. A signal and idler mode are picked out (for example, by means of diaphrams) and a low intensity laser beam with the same frequency as the idler photons is aligned with the idler mode from behind the crystal. The detector placed in the idler path is assumed to be able to measure photon number so that it can distinguish one photon from two photons, etc. Ou *et al.* [9] have also considered the arrangement in Fig. 1 (although they were interested in the case where the intensity of the laser behind the idler mode is high), and some of the treatment below is taken from their work. The interaction Hamiltonian for the parametric down-conversion process in the crystal is

$$\hat{H}_I = \left(i\hbar g \hat{a}_s^\dagger \hat{a}_i^\dagger V + \text{H.c.}\right), \tag{1}$$

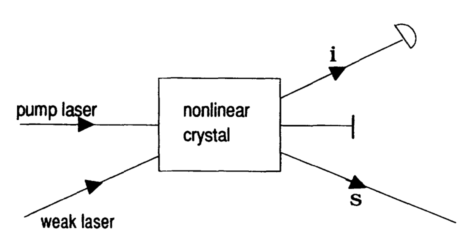

FIG. 1. Apparatus used to prepare the state $e|0\rangle + f|1\rangle$.

where g is a coupling constant, V is the complex amplitude of the pump (which we will treat classically), and $\hat{a}_s$ and $\hat{a}_i$ are the annihilation operators for the signal and idler modes, respectively. The initial state is $|\beta\rangle_i|0\rangle_s$, where $|\beta\rangle$ is a coherent state with amplitude β. After some short interaction time t, the state becomes

$$|\psi\rangle = \exp(-i\hat{H}_I t/\hbar)|\beta\rangle_i|0\rangle_s,$$

which can be expanded as

$$|\psi\rangle = |\beta\rangle_i|0\rangle_s + gtV\hat{a}_i^\dagger|\beta\rangle_i|1\rangle_s + \cdots. \quad (2)$$

The coherent state $|\beta\rangle$ can be written as

$$|\beta\rangle = e^{-\frac{1}{2}|\beta|^2}\sum_n \frac{\beta^n}{\sqrt{n!}}|n\rangle. \quad (3)$$

Substituting this into Eq. (2) and collecting terms, we obtain

$$|\psi\rangle = e^{-\frac{1}{2}|\beta|^2}[|0\rangle_i|0\rangle_s + |1\rangle_i(\beta|0\rangle_s + gtV|1\rangle_s) + |\varphi\rangle], \quad (4)$$

where the state $|\varphi\rangle$ contains only terms that have more than one photon in the idler mode. If we consider those times when only one photon is detected in the idler mode then we see from Eq. (4) that the state in the signal mode becomes $\beta|0\rangle_s + gtV|1\rangle_s$ which is in the required form.

The setup we will consider involves three 50:50 beam splitters. Consider a beam splitter with input modes a and b and output modes c and d. For such a beam splitter we can use the following transformations (see, for example, [10]),

$$|0\rangle_a|0\rangle_b \rightarrow |0\rangle_c|0\rangle_d, \quad (5a)$$

$$|0\rangle_a|1\rangle_b \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_c|1\rangle_d + i|1\rangle_c|0\rangle_d), \quad (5b)$$

$$|1\rangle_a|0\rangle_b \rightarrow \frac{1}{\sqrt{2}}(i|0\rangle_c|1\rangle_d + |1\rangle_c|0\rangle_d). \quad (5c)$$

For the sake of clarity, we will take it that the a mode is transmitted into the c mode. Then, the i factors in these transformations are picked up on reflection.

Now we come to the demonstration of nonlocality. The apparatus to be used is shown in Fig. 2. The state $q|0\rangle_s + r|1\rangle_s$ is prepared by the method described above and impinges onto one input (s) of a beam splitter. The other input (t) into this beam splitter is just the vacuum. The two outputs, u_1 and u_2, from this beam splitter each impinge onto a further beam splitter where they are each mixed with the coherent states $|\alpha_1\rangle_{a_1}$ and $|\alpha_2\rangle_{a_2}$, respectively. In the outputs from these beam splitters (c_1 and d_1 at end 1 and c_2 and d_2 at end 2) detectors are placed that can measure photon number. In addition, there are detectors that can be placed into paths u_1 and u_2. Let C_1 denote the detector in path c_1 and also let it denote the number of photons detected at that detector, such that if n photons are detected at detector C_1 then we will write $C_1 = n$. Similar notation is used for the other detectors.

The state impinging on the first beam splitter is $(q|0\rangle_s + r|1\rangle_s)|0\rangle_t$. Evolving through the beam splitter it becomes [using the transformations given in Eq. (5)]

$$|\Psi\rangle = q|0\rangle_{u_1}|0\rangle_{u_2} + \frac{ir}{\sqrt{2}}|1\rangle_{u_1}|0\rangle_{u_2} + \frac{r}{\sqrt{2}}|0\rangle_{u_1}|1\rangle_{u_2}. \quad (6)$$

There are two possible choices of measurement at each end: either with detector U_k in path u_k or with it removed. This makes a total of four possible experiments. We will now consider each of these experiments.

Experiment 1: Detectors U_1 and U_2 are put in paths u_1 and u_2, respectively. Since no more than one photon is emitted from the source at a time, it is clear that

$$U_1 = 1 \text{ and } U_2 = 1 \text{ never happens}. \quad (7)$$

This is also clear from Eq. (6) since there is no $|1\rangle_{u_1}|1\rangle_{u_2}$ term.

Experiment 2: Detector U_1 is removed from path u_1 and detector U_2 remains in path u_2. Thus, path u_1 is allowed to impinge on one input of the beam splitter at end 1 with the coherent state $|\alpha_1\rangle$ incident on the other input. The state of the system after the first beam splitter, given in Eq. (6), can be written

$$|\Psi\rangle = \left[q|0\rangle_{u_1} + \frac{ir}{\sqrt{2}}|1\rangle_{u_1}\right]|0\rangle_{u_2} + \left[\frac{r}{\sqrt{2}}|0\rangle_{u_1}\right]|1\rangle_{u_2}. \quad (8)$$

Now, consider the case in which no photons are detected at detector U_2 (so that $U_2 = 0$). When this happens, the state is projected onto the first term in Eq. (8) such that the state in mode u_1 becomes $N[q|0\rangle_{u_1} + \frac{ir}{\sqrt{2}}|1\rangle_{u_1}]$, where N is a normalization constant. Including the coherent state, the state entering the beam splitter at end 1 is

$$|\alpha_1\rangle_{a_1}N\left(q|0\rangle_{u_1} + \frac{ir}{\sqrt{2}}|1\rangle_{u_1}\right).$$

Expanding out the coherent state [using Eq. (3)] and collecting terms, we obtain

$$Ne^{-\frac{1}{2}|\alpha_1|^2}\left[q|0\rangle_{a_1}|0\rangle_{u_1} + \frac{ir}{\sqrt{2}}|0\rangle_{a_1}|1\rangle_{u_1} + \alpha_1 q|1\rangle_{a_1}|0\rangle_{u_1} + |\varphi'\rangle\right],$$

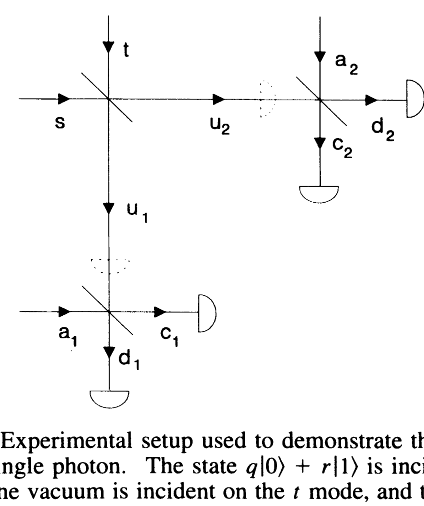

FIG. 2. Experimental setup used to demonstrate the nonlocality of a single photon. The state $q|0\rangle + r|1\rangle$ is incident on the s mode, the vacuum is incident on the t mode, and the coherent states $|\alpha_k\rangle$ are incident on the a_k modes ($k = 1,2$).

where $|\varphi'\rangle$ contains only terms with a total of two or more photons (as we shall only be interested in those cases in which a total of one photon is detected, we need not pay any special attention to the evolution of these terms). Evolving through the beam splitter this state becomes, using the transformations in (5),

$$Ne^{-\frac{1}{2}|\alpha_1|^2}\left[q|0\rangle_{c_1}|0\rangle_{d_1} + \left(\alpha_1 q - \frac{r}{\sqrt{2}}\right)|1\rangle_{c_1}|0\rangle_{d_1} + i\left(\alpha_1 q + \frac{r}{\sqrt{2}}\right)|0\rangle_{c_1}|1\rangle_{d_1} + |\varphi''\rangle\right], \quad (9)$$

where the state $|\varphi'\rangle$ has evolved to $|\varphi''\rangle$. We now take α to be set such that $\alpha_1 q + \frac{r}{\sqrt{2}} = 0$. This means that the two possibilities contributing to the $|0\rangle_{c_1}|1\rangle_{d_1}$ term will interfere destructively as can be seen from (9). Not forgetting that we are considering the case where $U_2 = 0$, we have the result that if $U_2 = 0$, then it never happens that $C_1 = 0$ and $D_1 = 1$. Thus, if we do find that $C_1 = 0$ and $D_1 = 1$, then we cannot have $U_2 = 0$ and so must have $U_2 = 1$ (since U_2 can only take values 0 and 1). If $C_1 = 0$ and $D_1 = 1$, then we will write $F_1 = 1$ for shorthand. Thus we have the relevant prediction of quantum theory for the second experiment:

$$\text{if } F_1 = 1 \text{ then } U_2 = 1. \quad (10)$$

Experiment 3: This experiment is the same as the previous experiment, but with 1 and 2 interchanged. α_2 is set such that if $U_1 = 0$, then there will be destructive interference of the two possibilities contributing to the $|0\rangle_{c_2}|1\rangle_{d_2}$ term. It is readily shown that the condition for this is $i\alpha_2 q + \frac{r}{\sqrt{2}} = 0$. This implies (by analogy with experiment 3) that

$$\text{if } F_2 = 1 \text{ then } U_1 = 1, \quad (11)$$

where $F_2 = 1$ is shorthand for $C_2 = 0$ and $D_1 = 1$.

Experiment 4: Detectors U_1 and U_2 are removed from paths u_1 and u_2, respectively. Thus, both paths impinge onto one input of a beam splitter with a coherent state incident on the other input in each. The prediction of quantum theory that is of interest in this experiment is

$$F_1 = 1 \text{ and } F_2 = 1 \text{ happens sometimes}. \quad (12)$$

To see that this is true, consider the evolution of the system. The state after the first beam splitter, including the coherent states, is $|\Psi\rangle|\alpha_1\rangle|\alpha_2\rangle$ where $|\Psi\rangle$ is given in Eq. (6). This can be expanded out using Eq. (3) to give

$$e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)}\left[\alpha_1\alpha_2 q|1\rangle_{a_1}|0\rangle_{u_1}|1\rangle_{a_2}|0\rangle_{u_2} + \frac{\alpha_1 r}{\sqrt{2}}|1\rangle_{a_1}|0\rangle_{u_1}|0\rangle_{a_2}|1\rangle_{u_2} + \frac{i\alpha_2 r}{\sqrt{2}}|0\rangle_{a_1}|1\rangle_{u_1}|1\rangle_{a_2}|0\rangle_{u_2} + |\varphi'''\rangle\right], \quad (13)$$

where $|\varphi'''\rangle$ contains all those terms that do not have a total of one photon at each end [and hence cannot have any relevance to the prediction (12)]. Upon evolving through the beam splitters at ends 1 and 2, the state becomes, using (5),

$$e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)}\left\{\left[\alpha_1\alpha_2\left(\frac{i}{\sqrt{2}}\right)^2 + \frac{\alpha_1 r}{\sqrt{2}}\frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i\alpha_2 r}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\right]|0\rangle_{c_1}|1\rangle_{d_1}|0\rangle_{c_2}|1\rangle_{d_2} + |\vartheta\rangle\right\}, \quad (14)$$

where all the terms not relevant to prediction (12) are contained in the state $|\vartheta\rangle$. Using the settings of α_1 and α_2 from experiments 3 and 4, we find that the square modulus of the coefficient in front of the $|0\rangle_{c_1}|1\rangle_{d_1}|0\rangle_{c_2}|1\rangle_{d_2}$ term is equal to $\frac{|r|^4}{16|q|^2}e^{-(|\alpha_1|^2+|\alpha_2|^2)}$. This is nonzero provided that q and r are nonzero, and hence prediction (12) follows.

To see how curious these predictions are consider the following scenario. Two observers, Alice at end 1 and Bob at end 2, each choose randomly whether to place their U_k detector in path u_k (in which case we will say that they are measuring U_k) or to remove it (in which case we will say that they are measuring F_k). Consider a run of the experiment in which Alice chooses to measure F_1 and Bob chooses to measure F_2, and the results $F_1 = 1$ and $F_2 = 1$ are obtained [this result sometimes follows from (12)]. From her result $F_1 = 1$ and the prediction (10), Alice can deduce that if there is a detector placed in the u_2 path at the other side then a photon will be detected there, and from this she may deduce that the photon from the source actually went towards Bob (at least in the sense that if a detector had been placed in path u_2 it would have fired). On the other hand, from his result $F_2 = 1$ and prediction (11), Bob can deduce that if there is a detector placed in path u_1 then a photon will be detected there, and from this he may deduce that the photon from the source actually went towards Alice (at least in the sense that if a detector had been placed in path u_1 it would have fired). However, there is, at most, only one photon emitted from the source, so they cannot both be right. That is to say, if detectors had been placed in paths u_1 and u_2 then, for this particular run of the experiment, both detectors would have fired—but this violates prediction (7). Thus we have a contradiction. This argument is so persuasive that one might be tempted to think that quantum mechanics must be wrong. However, we will now see that there is an implicit assumption of locality in this reasoning, and that without this assumption there is no contradiction. Alice obtains $F_1 = 1$. Bob is actually measuring F_2. Alice might deduce from her result and the prediction (10) that had Bob measured U_2 instead he would have gotten $U_2 = 1$. However, without assuming locality, this deduction is wrong, because if Bob had decided to measure U_2 instead, there might then have been a nonlocal influence from Bob's end to Alice's end, and Alice might not then have obtained the result $F_1 = 1$ [so then she could not use the result (10) to deduce that $U_2 = 1$]. Of course, similar remarks apply to Bob. (See [8,11] for more rigorous discussion on how predictions of this form violate locality.)

Some further remarks relating to the above calculations are necessary. If the idler detector clicks at time t, we require that the given state has been emitted within some time interval $t - \tau/2$ to $t + \tau/2$ where $c\tau$ is small compared with the dimension of the experiment (the idler detector clicking acts as a trigger for the other detectors). It follows from the energy-time uncertainty inequality that there must be a certain spread of frequency in the photons emitted from the source. Consequently, a full treatment requires a multimode calculation. Thus, rather than $q|0\rangle + r|1\rangle$, the state incident on the s input is

$$q|0\rangle_s + \sum_i r_i|1\rangle_{s_i}, \tag{15}$$

where i labels different frequencies (as usual, the sum over frequencies turns into a integral as the mode spacing tends to zero). The source photons must be indistinguishable from the photons emitted from the local oscillators used to make the measurements at ends 1 and 2, and therefore the latter must also have the same spread of frequency. Thus, let these local oscillators be described by multimode coherent states $|\{\alpha_k\}\rangle = \prod_i |\alpha_k^i\rangle$. Expanding out the fist few terms in this product [using a similar equation to (3)] we get

$$|\{\alpha_k\}\rangle = e^{\left(-\frac{1}{2}\sum_i|\alpha_k^i|\right)}\left(|0\rangle + \sum_i \alpha_k^i|1\rangle_i + \cdots\right). \tag{16}$$

The previous calculations can be easily repeated with the multimode states Eqs. (15) and (16). Thus, $r|1\rangle_s$ is replaced with $\sum_i r_i|1\rangle_{s_i}$ and $\alpha_k|1\rangle_{a_k}$ is replaced with $\sum_i \alpha_k^i|1\rangle_{ia_k}$ and so on through the calculation. The predictions (7), (10), (11), and (12) that were used to run the argument against nonlocality are recovered if we set the complex amplitude of each frequency mode in the multimode coherent states according to

$$\alpha_1^i = i\alpha_2^i = -\frac{r_i}{\sqrt{2}q}. \tag{17}$$

In practice this would be very difficult to achieve. However, in principle, it could be done by taking a very large number of single mode lasers and combining them at a multiport beam splitter in such a way as to produce the required multimode coherent state.

Given the difficulties in realizing such an experiment, it is interesting to look at the case where we deviate slightly from the inequality-free proof of nonlocality. It can be readily verified that the Clauser-Horne-Bell inequalities [12] can be put into the form

$$\begin{aligned}-1 \le \text{Prob}(F_1 = 1\&F_2 = 1) &- \text{Prob}(F_1 = 1\&U_2 = 0)\\ &- \text{Prob}(U_1 = 0\&F_2 = 1)\\ &- \text{Prob}(U_1 = 1\&U_2 = 1) \le 0\end{aligned} \tag{18}$$

(where here F_k and U_k can refer to any measurements that might be performed at end k of an apparatus). With the predictions (7), (10), (11), and (12) the first probability in (18) is equal to $\varepsilon_1\varepsilon_2|\mu\nu|^4N^2$ and the other probabilities are all equal to zero, and so the upper limit inequality is violated. If we deviate slightly from these predictions, then as long as the deviation is not too great, this inequality will still be violated, and so we still have nonlocality.

In conclusion, we see that the original worries about nonlocality of a single particle expressed by Einstein at the 1927 Solvay Conference have some solid basis in fact. One natural question that arises is whether this nonlocality is only restricted to photons. The answer would seem to be that an analogous proof to the above could be constructed for any type of particle for which it is possible to prepare a direct superposition of that particle with the vacuum. However, for a vast range of types of particles there are superselection rules that prohibit just exactly this, and nonlocality with single particles of this type could not be observed.

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