Radial Structure of High-Mode-Number Toroidal Modes in General Equilibrium Profiles

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The problem of global radial envelope structure of high-mode-number toroidal modes in general equilibrium profiles is studied using the coordinate transformation formalism. It is shown that there exists a continuum of toroidal eigenmodes with well-localized radial structure.

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It is now well recognized that in toroidal plasma systems toroidal modes with ballooning structure can be generated from the coupling of cylindrical modes. The well-known ballooning representation formalism (BRF) [1] has served as a powerful tool to analyze the stability of these toroidal modes in the high-mode-number regime. This formalism assumes initially a particular eigenfunction form, called the "ballooning representation." To the lowest order of $1/n$, where n is the toroidal mode number, complicated two-dimensional (2D) eigenvalue problems then reduce to simple one-dimensional (1D) problems, which can be easily solved. A number of works have used this lowest order ballooning analysis to estimate the stability of various high-n toroidal instabilities in axisymmetric toroidal plasma systems. An important problem of the lowest order ballooning analysis is, however, that the lowest order solutions are not localized radially but just propagate. To see whether they localize radially well, we should consider the next order equations. This problem, known as the next order radial envelope profile (REP) problem, has been previously considered near a maximum in the equilibrium quantities like the diamagnetic frequency ω_*, showing that there can exist a toroidal mode (more specifically, the mode with $\theta_0 = 0$ among the lowest order solutions over $0 < \theta_0 < 2\pi$, where θ_0 is the Bloch shift parameter) with well-localized radial structure in this case [1]. However, in the more general case, for example, at the plasma radius with a linear equilibrium variation which actually covers most of plasma region, it is still not clear whether a similarly localized mode can exist or not, even though numerous previous works performed the lowest order stability estimates, assuming implicitly the existence of such a mode (at $\theta_0 = 0$).

In this Letter, we consider this problem of the global radial structure of high-n toroidal modes in the more general case. For this study, we employ a new formalism, the coordinate transformation formalism (CTF), similar to the "twisted slice or eddy" model [2,3] instead of the usual BRF. While the BRF is physically insightful, it has some complication and difficulty in treating the next order REP problem, mainly due to the usage of the assumed eigenfunction form. The present CTF does not use this assumption but solves the 2D eigenvalue equation almost straightforwardly. This enables us to handle more clearly and easily the next order REP problem.

We show that in the more general equilibrium profile case there also exist toroidal eigenmodes with well-localized radial structure. These new modes which can now arise over most of the plasma region differ from the conventional mode arising near the maximum equilibrium point, in that they exist as a continuum and have the asymmetric shapes poloidally and radially. However, they have the similar magnitudes of radial scale length and maximum growth rate with the conventional mode, and thus implying that numerous previous works, estimated the stability of toroidal modes in general equilibrium profiles assuming the existence of the conventional type mode, might be almost acceptable. How these new continuum modes are related to the mode, found recently in Ref. [4] from the BRF, is also discussed.

We start our study from the general form of a 2D eigenvalue equation in the usual (r, θ) coordinate system:

$$L\left(\omega, r, \theta, \vec{\nabla}_\perp, \nabla_\parallel\right)\tilde{\phi} = 0, \tag{1}$$

where $\vec{\nabla}_\perp = \partial/\partial r\hat{r} + (1/r)\partial/\partial\theta\hat{\theta}$ and $\nabla_\parallel = \vec{b}\cdot\vec{\nabla} = (1/Rq)(\partial/\partial\theta + q\partial/\partial\xi)$. Following the standard procedure, we solve this equation order by order in terms of the parameter $\delta = 1/n \ll 1$. Assuming the eigenmode is centered at the rational surface $r = r_0$, where $nq(r_0) = m$, writing the perturbed function $\tilde{\phi}$ in the form $\tilde{\phi}(r,\theta) = \hat{\phi}(r,\theta)e^{-i(n\xi - m\theta)}$ and then expanding all equilibrium quantities around r_0, like $nq(r) = nq(r_0) + sk_y x + nq''(r_0)x^2/2\ldots$, where $x \equiv r - r_0$, $s \equiv r_0 q'(r_0)/q(r_0)$, and $k_y \equiv m/r_0$, we can obtain first the lowest order equation

$$L_0\left(\omega_0, \theta, \frac{\partial}{\partial x}, \frac{\partial}{\partial \theta} - isk_y x\right)\hat{\phi}_0(x,\theta) = 0. \tag{2}$$

We note that to the lowest order the radial variation comes through only the magnetic shear term $sk_y x$. We will first solve this lowest order 2D equation (2) using the CTF (the subscripts "0" are dropped for a moment). A clear mathematical definition of usual ballooning space variables and ballooning equations will be obtained during this solving process, and also it will be clarified why the next order REP problem occurs.

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Lowest order solution by the CTF.—The essential part of the CTF is just to take the coordinate transformation

$$\theta' = \theta\,, \qquad k_x' = k_x - sk_y\theta\,, \tag{3}$$

where k_x is the Fourier transform of x:

$$\hat{\phi}(x,\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_x\, e^{ik_x x}\Phi(k_x,\theta)\,. \tag{4}$$

It is easy to see that the lowest order 2D equation (2) then reduces to the following 1D form:

$$L\left(\omega,\theta,\theta_0 - \theta,\frac{\partial}{\partial\theta}\right)\Phi(\theta) = 0\,, \tag{5}$$

with taking $\theta_0 \equiv -k_x'/sk_y$ as a good quantum number.

We first note that the reduced 1D equation (5) is exactly the same form as the usual ballooning equation. This means that the ballooning equation can be also derived by a coordinate transformation, as also noted in Refs. [2] and [3], instead of using the ballooning representation. A novel advantage of the above simple derivation is that it enables us obtain the following two mathematical pictures: (i) the Bloch shift parameter θ_0 can be seen as a coordinate in the coordinate system where the lowest order 2D equation reduces to the 1D form; and (ii) the ballooning equation can be seen as a differential equation along the characteristic line $\theta_0 = -k_x'/sk_y = \theta - k_x/sk_y$ in the (k_x,θ) space. As will be shown, these two pictures become very useful in understanding how the present CTF is related to the conventional BRF and also why the next order REP problem occurs.

To complete the lowest order problem, we need further to specify proper boundary conditions to solve the reduced 1D equation (5). Obviously, any physical solution in the 2D (k_x,θ) space should satisfy the two general boundary conditions: (a) $\Phi(k_x,\theta) \to 0$ as $|k_x| \to \infty$ and (b) the periodicity along θ. Here, it is easy to see, noting the above picture (ii), that the first condition (a) can be satisfied well if we require $\Phi_{|\theta|\to\infty}(\theta) = 0$ for Eq. (5). We note that the variable θ has thus changed its meaning from a usual poloidal angle to the extended ballooning angle $(-\infty < \theta < \infty)$. On the other hand, to satisfy the second condition (b), we use the periodic property of eigenvalue $\omega(\theta_0) = \omega(\theta_0 + 2\pi l)$, $l = \pm 1, \pm 2, \ldots$, which is obvious from Eq. (5) where θ_0 dependence comes only in the form of $(\theta_0 - \theta)$. This property implies the existence of infinite numbers of shifted functions $\Phi_{\theta_0+2\pi l}(\theta) = \Phi_{\theta_0}(\theta - 2\pi l)$, with the same eigenvalue. If we now make the following function:

$$\bar{\Phi}_{\theta_0}(\theta) = \sum_{l=-\infty}^{\infty}\Phi_{\theta_0+2\pi l}(\theta) = \sum_{l=-\infty}^{\infty}\Phi_{\theta_0}(\theta - 2\pi l)\,, \tag{6}$$

we can see that $\bar{\Phi}_{\theta_0}$ then becomes periodic along θ. Thus, we find that the two general boundary conditions (a) and (b) can be satisfied well, if we solve Eq. (5) with the condition $\Phi_{|\theta|\to\infty}(\theta) = 0$ and then take the infinite sum (6). Basically, this solving procedure is similar with that in the usual BRF, but note that we obtained here the results without assuming any eigenfunction form.

Next order REP problem.—The above lowest order solution (6), satisfying the two boundary conditions (a) and (b), looks complete by itself. One problem is, however, found if we now try to draw the eigenfunction shape in the 2D (k_x,θ) space, i.e., we see that the above lowest order solution determines the eigenfunction shape, only along the characteristic line $\theta_0 = \theta - k_x/sk_y =$ const in the 2D (k_x,θ) space. Clearly, in drawing a complete 2D profile this is insufficient. We need further to know the eigenfunction shape along the other characteristic line or along θ_0, and this is the basic reason why we should consider the next order equations.

Before we try to determine the shape along θ_0 from the next order equations, it is interesting to see first what shape along θ_0 is physically desirable. Let us first assume the shape along θ_0 by the δ function $\delta(\theta_0 - \Theta_0)$ or $\delta(k_x - sk_y(\Theta_0 + 2\pi l - \theta))$. With this assumption, the mode will have a highly localized profile around the $\theta_0 = \Theta_0$ along θ_0, with the eigenvalue $\omega(\Theta_0)$. The eigenfunction form in the (k_x,θ) space then becomes $\bar{\Phi}_{\Theta_0}(k_x,\theta) = \sum_{l=-\infty}^{\infty}\Phi_{\Theta_0}(\theta - 2\pi l)\delta(k_x - sk_y(\Theta_0 + 2\pi l - \theta))$ from Eq. (6), and taking the inverse Fourier transform of this we can obtain

$$\hat{\phi}(x,\theta) = \sum_{l=-\infty}^{\infty}\Phi_{\Theta_0}(\theta - 2\pi l)e^{isk_y x(\theta - 2\pi l - \Theta_0)}. \tag{7}$$

[We note that the solution (7) is exactly the same form as the well-known ballooning representation. From the present CTF approach, we thus obtain the ballooning representation as an approximate solution of the lowest order equation, under the δ-function assumption.] It is easily observed that the solution (7) does not converge radially but just propagates. This means that the δ-function shape along θ_0 corresponds to an unphysical solution with infinite radial width. To obtain a more physical solution, let us now assume the other shape $e^{-(\theta_0-\Theta_0)^2/\epsilon}$ with $\epsilon \to 0$, instead of the δ function. We can see that the solution (7) then becomes proportional to $e^{-\epsilon x^2}$ so that the solution converges radially. Thus, we find that the solution, with a highly localized but broader shape than the δ function along θ_0, is desirable from the next order equation for the well-localized global radial structure.

The next order equations can be basically obtained from Eq. (1), by taking the next order terms. First, for the first order term we note that there exist several sources to contribute: (α) the equilibrium variation in $T_e(r)$, $T_i(r)$, $q(r)$, etc.; (β) the variation of radial variable r itself from r_0; and (γ) the poloidal differential operator $\partial/\partial\theta$ in terms of $\hat{\phi}$. The terms x/L_{T_e}, x/r_0, and $(1/m)\partial/\partial\theta$, which come from the above sources (α), (β), (γ), respectively, where $L_{T_e} = -(d\ln T_e/dr)^{-1}$, change to $i\delta(r_0/sL_{T_e})\partial/\partial\theta_0$, $i\delta(1/s)\partial/\partial\theta_0$, and $i\delta\partial/\partial\theta_0$ (note $\partial/\partial\theta \to \partial/\partial\theta|_{\theta_0} + \partial/\partial\theta_0 \sim \partial/\partial\theta_0$ since $\partial/\partial\theta_0 \gg \partial/\partial\theta|_{\theta_0} \sim 1$ with the expected highly localized shape along θ_0), under the coordinate transformation (3).

All these terms can have the same magnitude of order $\delta\partial/\partial\theta_0$ ($\sim\delta^{1/2}$ as will be shown) in the typical case of $r_0 \sim L_{T_e}$ and $s \sim 1$. This illustrates the important fact that the first order term will be finite in general from the sources (β) and (γ), even when there is no equilibrium variation, unlike the usual assumption from the BRF [1]. We can thus expect that the shape along θ_0 or the REP will be determined in general from the first order equation, and from now on we consider this first order REP problem.

Including the above first order terms, the 2D eigenvalue equation (1) can be written in a form, in the (θ, θ_0) space,

$$\left[L_0\left(\omega, \theta, \theta_0, \frac{\partial}{\partial\theta}\right) + i\delta L_1\left(\omega, \theta, \theta_0, \frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta_0}\right]\Phi = 0. \tag{8}$$

Here, the first term represents the lowest order operator which, as already shown, determines the lowest order eigenvalue $\omega_0(\theta_0)$ and eigenfunction Φ_{θ_0} at each θ_0. The second term, proportional to $\partial/\partial\theta_0$, is the first order contributions from the above sources (α), (β), and (γ). This term couples the lowest order solutions along θ_0, determining the profile along θ_0. From the earlier discussion, we expect a highly localized profile along θ_0 around $\theta_0 = \Theta_0$ from this coupling and assume *a priori* $\partial/\partial\theta_0 \sim \delta^{-1/2} \sim n^{1/2} \gg 1$. The second term in Eq. (8) then becomes order $\delta^{1/2}$, and we can expand to $\Phi = \Phi_0 + \delta^{1/2}\Phi_1$ and $\omega = \omega_0(\Theta_0) + \delta^{1/2}\omega_1$. Now, assuming the lowest order eigenfunction Φ_0 has the form $\Phi_0 = A(\theta_0)\Phi_{\theta_0}$ with the envelope function $A(\theta_0)$ and, expanding the L_0 and Φ_{θ_0} around $\theta_0 = \Theta_0$, we can obtain the following first order equation:

$$\left(\delta^{1/2}\omega_1 - \frac{\partial\omega_0}{\partial\theta_0}\theta_0'\right)\frac{\partial L_0}{\partial\omega}A(\theta_0')\Phi_{\Theta_0} + i\delta L_1\frac{\partial A(\theta_0')}{\partial\theta_0'}\Phi_{\Theta_0} + \delta^{1/2}L_0\Phi_1 = 0, \tag{9}$$

where $\theta_0' \equiv \theta_0 - \Theta_0$. Multiplying Φ_{Θ_0} and averaging over θ, this then yields the envelope equation

$$C_0\left(\delta^{1/2}\omega_1 - \frac{\partial\omega_0}{\partial\theta_0}\theta_0'\right)A + i\delta C_1\frac{\partial A}{\partial\theta_0'} = 0, \tag{10}$$

of which the solution is

$$A(\theta_0) = \exp\left[\frac{i}{\delta}\frac{C_0}{C_1}\left(\delta^{1/2}\omega_1\theta_0' - \frac{1}{2}\frac{\partial\omega_0}{\partial\theta_0}(\theta_0')^2\right)\right], \tag{11}$$

where $C_0 = \int_{-\infty}^{\infty} d\theta\Phi_{\Theta_0}(\partial L_0/\partial\omega)\Phi_{\Theta_0}$ and $C_1 = \int_{-\infty}^{\infty} d\theta\Phi_{\Theta_0}L_1\Phi_{\Theta_0}$.

We first note that the envelope (11) can have a well-localized profile along θ_0 with the self-consistent localization width $\Delta\theta_0' \sim \delta^{1/2}$, if (I) $\omega_1 = 0$ and (II) $\mathrm{Im}[-(C_0/C_1)\partial\omega_0/\partial\theta_0] > 0$. Here, the second condition (II) can be written in a more precise form,

$$\mathrm{Im}\left(-\frac{C_0}{C_1}\frac{\partial\omega_0}{\partial\theta_0}\right) \geq \left|\mathrm{Im}\left(\frac{C_0}{3C_1}\frac{\partial^2\omega_0}{\partial\theta_0^2}\right)\right|, \tag{12}$$

if we require the second derivative term, in the expansion $\omega_0(\theta_0) = \omega_0(\Theta_0) + (\partial\omega_0/\partial\theta_0)\theta_0' + (1/2)(\partial^2\omega_0/\partial\theta_0^2)(\theta_0')^2 + \cdots$, to be negligible to the order $\delta^{1/2}$, compared with the first derivative term in Eq. (11).

The condition (12) is the main result of this work. It defines an acceptable range in the θ_0 values for the lowest order solution to have a well-defined radial structure in the finite first order case of $C_1 \sim 1$. [In contrast, in the limit of $C_1 \sim 0$ Eq. (10) gives just $\omega_1 = 0$ and $\partial\omega_0/\partial\theta_0 = 0$, so that we should consider the second order equation to determine the envelope. This is exactly the conventional case considered in the previous work [1].] For example, in the limit where C_0 and C_1 are dominantly real, the condition (12) is just related to the lowest order growth rate $\gamma_0(\theta_0)$. For the typical case with $\gamma_0(\theta_0) \sim \cos\theta_0$, the condition (12) then gives the range $\sin\theta_0 \geq |\cos\theta_0|/3$ or $0.32 \leq \theta_0 \leq \pi - 0.32$. As far as θ_0 is within this range, the zeroth order eigenvalue is correct to the order $\delta^{1/2}$ from the condition (I), and the eigenfunctions have the well-localized radial structure with $\Delta\theta_0 \sim \delta^{1/2}$ or $\Delta x \sim \rho_s\delta^{-1/2} \sim (\rho_s r_0)^{1/2}$. In other words, this result means that in a general equilibrium profile there can exist a continuum of high-n toroidal modes with well-localized radial structure. The maximum growth rate (~ 0.9 at $\theta_0 = 0.32$ for the above example) of the continuum is a little smaller than the usual lowest order estimate (1 at $\theta_0 = 0$) based on the conventional mode. Also, the continuum modes have asymmetric shapes poloidally and radially, since $\theta_0 \neq 0$.

Besides the above general case, for the comparison with previous works [4,5] it is interesting to consider a simple case where the operators L_0 and L_1 have the particular forms

$$L_0 = L_0^0\left(\theta, \theta_0, \frac{\partial}{\partial\theta}\right) + L_0^1(\omega), \quad L_1 = L_1(\omega, \theta_0). \tag{13}$$

Note that in L_0 the eigenvalue part is well separated while L_1 has no θ dependence. In this particular case, Eq. (8) can be solved quite simply. We need just to assume the eigenfunction has the form $\Phi = A(\theta_0)\Phi_{\theta_0}$ with $L_0\Phi_{\theta_0} = 0$. Then Eq. (8) becomes

$$[L_0^1(\omega) - L_0^1(\omega_0)]A(\theta_0)\Phi_{\theta_0} + i\delta L_1(\omega, \theta_0)\left[\frac{\partial A}{\partial\theta_0}\Phi_{\theta_0} + A\frac{\partial\Phi_{\theta_0}}{\partial\theta_0}\right] = 0, \tag{14}$$

which is just the equation of $A(\theta_0)$ only (note the last term $A\partial\Phi_{\theta_0}/\partial\theta_0$ can be neglected by the highly localized envelope assumption), giving the solution

$$A(\theta_0) = e^{\frac{i}{\delta}\int^{\theta_0} f(\omega, \theta_0')d\theta_0'}, \tag{15}$$

where $f(\omega, \theta_0') = [L_0^1(\omega) - L_0^1(\omega_0(\theta_0'))]/L_1(\omega, \theta_0')$.

We first note that the envelope (15) can have a well-localized profile around $\theta_0 = \Theta_0$ with $\omega = \omega_0(\Theta_0)$, if $\mathrm{Im}(\partial f/\partial\theta_0) \geq |\mathrm{Im}(\partial^2 f/3\partial\theta_0^2)|$ at $\theta_0 = \Theta_0$, which is exactly the same condition as Eq. (12).

On the other hand, let us try to impose the condition

$$\frac{1}{\delta}\int_0^{2\pi} f(\omega,\theta_0')d\theta_0' = 2\pi n, \quad n = 0,\pm 1,\pm 2,\ldots, \tag{16}$$

which makes $A(\theta_0)$ periodic along θ_0. If we use the earlier picture (ii) of the ballooning equation, we can see that the eigenfunction $\Phi(\theta,\theta_0) = A(\theta_0)\Phi_{\theta_0}(\theta,\theta_0)$ then becomes periodic along θ by itself in the (k_x,θ) space, without the necessity to make the infinite series (6). This solution, which seems to have a different form from the typical solution (7), is indeed that found recently in Ref. [4] from the usual BRF. From the present CTF approach we can obtain a more clear understanding of this solution, and here we add some important comments.

We first note that this solution would be essentially the same type as the conventional mode with the infinite series form. This is because with $\delta \ll 1$ the envelope (15) is highly peaked at the maximum amplitude points $\theta_0 = \Theta_0 + 2\pi l$ with the localization width of order $\delta^{1/2}$, where $l = 0,\pm 1,\ldots$ [the maximum amplitude point Θ_0 can be obtained by the conditions $\mathrm{Im} f(\omega,\theta_0) = 0$ and $\partial \mathrm{Im} f/\partial\theta_0 > 0$, with ω determined from Eq. (16)], and then taking only these parts near the maximum points to the order $\delta^{1/2}$ the solution has the infinite series form. Furthermore, since the localization width is $\Delta\theta_0 \sim \delta^{1/2}$, the solution has the radial width $\Delta x \sim \rho_s\delta^{1/2} \sim (\rho_s r_0)^{1/2}$, which is also similar with the conventional mode. With these observations, we can now say that the above solution is nothing but one within the continuum given in Eq. (12). In fact, considering the simple case with $L_0^1 = \omega$, $L_1 = 1$, and $\gamma_0(\theta_0) = \cos\theta_0$, the condition (16) gives $\gamma = 0$ with $\Theta_0 = \pi/2$, which clearly belongs to just one among the continuum (12) which has $0.9 < \gamma < -0.9$ with $0.32 \le \Theta_0 \le \pi - 0.32$. The condition (16), which is obtained in Ref. [4] using the usual WKB argument, permits only one solution because the condition (16), indeed, requires the well behavior of the envelope $A(\theta_0)$ to the arbitrary order in δ over all of the region $0 < \theta_0 < 2\pi$. If we, however, relax this requirement to the order $\delta^{1/2}$ consistent with the first order treatment, we can obtain the more flexible condition (12) allowing more solutions and a larger maximum growth rate.

A more physical understanding of how the continuum modes can exist in the general equilibrium profile case may be obtained by considering the eigenfunction shape. For the case near the maximum, the global equilibrium variation is symmetric around the maximum point, and the mode which can incorporate well this symmetric equilibrium variation should be the mode, which is symmetric to the lowest order, and thus, only the mode with $\theta_0 = 0$ [1]. In contrast, for the more general case (like the linearly varying equilibrium profile) where the equilibrium variation is asymmetric, there exists a *continuum* of modes ($0 < \theta_0 < \pi$), which are asymmetric to the lowest order and thus can incorporate well the asymmetric equilibrium variation. We can thus expect the more solutions (12) in this case.

The result of this work, showing the existence of a continuum of high-n toroidal modes in general equilibrium profile, implies that the toroidal modes, like the toroidal η_i or the ballooning modes, might be actually playing important roles in real plasma systems. It also suggests that numerous previous works, estimated the stability of various high-n toroidal modes based on the conventional mode picture, might be almost acceptable.

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