General Relativistic Gravitational Field of a Rigidly Rotating Disk of Dust: Axis Potential, Disk Metric, and Surface Mass Density

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In a recent paper we described the general relativistic gravitational field of a rigidly rotating disk of dust in terms of two linear integral equations (a "small" one and a "big" one). Here we present the exact solution to the small integral equation. The Ernst potential on the symmetry axis, the disk metric, as well as the surface mass density are given up to quadratures in terms of the $\wp$ function of the Weierstrass.

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Recently the solution describing the general relativistic gravitational field of the rigidly rotating disk of dust has been found [1] by solving a boundary value problem, first formulated and attacked by Bardeen and Wagoner [2,3]. This solution is a *global* one, i.e., it represents the "exterior" as well as the "interior" field. Apart from its astrophysical importance the model might be a first step toward the exact solution of rotating body problems in general relativity. A remarkable feature of the exterior solution is the fact that, in a certain parameter limit, it approaches exactly the extreme Kerr solution.

Our solution was derived by applying the inverse (scattering) method. As a consequence, the metric has to be calculated from the solution of a linear integral equation (the "big" integral equation). The kernel of that equation may be found from the boundary conditions. This task leads us to another linear integral equation, the "small" one [see Eq. (13) below]. Its analytic solution is given in the present Letter. This allows us to calculate explicitly the Ernst potential on the symmetry axis, the disk metric, as well as the surface mass density. The details of the derivation of both integral equations and the related solution techniques will be described in a subsequent paper.

The solution of the problem may be formulated in Weyl-Lewis-Papapetrou coordinates:

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a d\varphi)^2. \qquad (1)$$

(Throughout the paper we use units where Newton's gravitational constant G as well as the velocity of light c are equal to 1.) The metric functions $e^{2U}(\rho,\zeta)$, $e^{2k}(\rho,\zeta)$, and $a(\rho,\zeta)$ depend uniquely on two parameters, Ω and μ. Ω is the angular velocity of the disk as seen by an observer at infinity. The parameter μ introduced in [1] is related to the "surface potential" $V_0 \equiv U(\rho = 0, \zeta = 0)$ and the coordinate radius ρ_0 of the disk:

$$\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}. \qquad (2)$$

[Note that V_0 also determines the relative central redshift $z_0 = \exp(-V_0) - 1$ of the disk measured by an observer at infinity.]

It turns out that V_0 is a function of μ alone; $V_0 = V_0(\mu)$. All the results of the present paper may be expressed in terms of that function $V_0(\mu)$.

The parameter function $V_0(\mu)$.—As a consequence of the small integral equation, the dependence of V_0 on μ is given by the following expression:

$$\sinh 2V_0 = -\mu - \frac{1+\mu^2}{\wp[I(\mu); \frac{4}{3}\mu^2 - 4, \frac{8}{3}\mu(1 + \frac{\mu^2}{9})] - \frac{2}{3}\mu}, \qquad (3)$$

where

$$I(\mu) = \frac{1}{\pi}\int_0^{\mu} \frac{\ln(x + \sqrt{1+x^2})\,dx}{\sqrt{(1+x^2)(\mu - x)}}, \qquad (4)$$

and $\wp$ is the Weierstrass function defined by

$$\int_{\wp(x;g_2,g_3)}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = x. \qquad (5)$$

The parameter range $0 > V_0 > -\infty$, with $|V_0| \ll 1$ being the Newtonian limit, corresponds to

$$0 < \mu < \mu_0 = 4.629\,661\,843\,474\,342\,042\,6\ldots, \qquad (6)$$

where μ_0 is the first zero of $\wp[I(\mu); \frac{4}{3}\mu^2 - 4, \frac{8}{3}\mu(1 + \frac{\mu^2}{9})] - \frac{2}{3}\mu$. The limit $\mu \to \mu_0$ leads, for $\rho^2 + \zeta^2 \neq 0$, to the extreme Kerr solution (see Ref. [1]). The physical meaning of the solution for $\mu > \mu_0$ is under investigation.

Note that from $V_0(\mu)$ one obtains $\rho_0(\Omega, \mu)$ according to (2). In the next paragraphs we will show that, in a sense, the whole solution of the problem may be displayed from $V_0(\mu)$.

The Ernst potential on the symmetry axis.—The (complex) Ernst potential is defined by

$$f = e^{2U} + ib, \qquad (7)$$

where $b(\rho,\zeta)$ is related to the metric function $a(\rho,\zeta)$ according to

$$b_{,\rho} = -\frac{e^{4U}}{\rho} a_{,\zeta}, \qquad b_{,\zeta} = \frac{e^{4U}}{\rho} a_{,\rho}. \qquad (8)$$

The symmetry of our problem implies

$$f(\rho, -\zeta) = \bar{f}(\rho, \zeta), \qquad (9)$$

where the bar denotes complex conjugation.

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For our solution [1] the Ernst potential on the symmetry axis $\rho = 0$ is given by

$$f(\rho = 0, \zeta > 0) = \frac{2\pi + \int_{-1}^{1} \frac{\beta(x)dx}{ix - \zeta/\rho_0}}{2\pi + \int_{-1}^{1} \frac{\alpha(x)dx}{ix - \zeta/\rho_0}}, \tag{10}$$

where $\alpha(x)$ algebraically depends on $\beta(x)$ and x:

$$\alpha(x) = \frac{(1 - w)\beta + i\sqrt{4w^2 e^{-4V_0}(b_0^2 + 4\Omega^2\rho_0^2 x^2) - (e^{4V_0} + w^2)\beta^2}}{ib_0 - 2\Omega\rho_0 x}, \tag{11}$$

with

$$w = 2\Omega^2\rho_0^2(1 - x^2), \qquad b_0 = -\sqrt{1 - e^{4V_0} - 4\Omega^2\rho_0^2}, \tag{12}$$

and β has to satisfy the small integral equation:

$$\beta(x) = (2\mu)^{3/2} e^{-V_0} x(1 - x^2) - \mu^2 \left[(1 - x^2)^2 \beta(x) + \frac{1 - x^2}{\pi^2} ⨍_{-1}^{1} dx' \frac{1 - x'^2}{x' - x} ⨍_{-1}^{1} dx'' \frac{\beta(x'')}{x'' - x'} \right], \tag{13}$$

where $⨍$ denotes Cauchy's principal value. Here we present the exact solution:

$$\beta(x) = 2Ce^{-V_0}\sqrt{\frac{-2(\sinh 2V_0 + C)}{1 + C^2}} \sin\chi, \tag{14}$$

with

$$C = \mu(1 - x^2) = we^{-2V_0} \tag{15}$$

and

$$\chi(x) = 2x\sqrt{\mu(1 + C^2)} \times \int_0^1 \frac{V_0'(\tilde{\mu})\sqrt{-[\sinh 2V_0(\tilde{\mu}) + \tilde{\mu}]}}{\sinh 2V_0(\tilde{\mu}) + C} dy,$$
$$\tilde{\mu} = C + \mu x^2 y^2. \tag{16}$$

The notation $V_0(\tilde{\mu})$ indicates that the argument μ in the parameter function $V_0(\mu)$ defined by Eq. (3) has to be replaced by $\tilde{\mu} = C + \mu x^2 y^2$. $V_0'(\tilde{\mu})$ means $dV_0(\tilde{\mu})/d\tilde{\mu}$.

Disk metric and surface mass density.—The metric functions in the disk (i.e., for $\zeta = 0, \rho \le \rho_0$) are given by the following expressions:

$$e^{2U} = \exp\{2V_0(\mu[1 - \rho^2/\rho_0^2])\} - \frac{\mu\rho^2}{2\rho_0^2}, \tag{17}$$

$$(1 + \Omega a)e^{2U} = \exp\{V_0(\mu)\}\exp\{V_0(\mu[1 - \rho^2/\rho_0^2])\}, \tag{18}$$

and

$$e^{2k-2U} = \exp\{-2V_0(\mu)\} \times \exp\left\{ -\int_{\mu(1-\rho^2/\rho_0^2)}^{\mu} \frac{f_0'(\tilde{\mu})\bar{f}_0'(\tilde{\mu})}{f_0(\tilde{\mu}) + \bar{f}_0(\tilde{\mu})} d\tilde{\mu} \right\}. \tag{19}$$

Again, $V_0(\mu[1 - \rho^2/\rho_0^2])$ means that the argument μ in $V_0(\mu)$ must be replaced by $\mu(1 - \rho^2/\rho_0^2)$. In Eq. (19) $f_0(\tilde{\mu})$ is defined by the replacement $\mu \to \tilde{\mu}$ in the complex parameter function

$$f_0(\mu) = e^{2V_0(\mu)} + ib_0(\mu), \tag{20}$$

representing the Ernst potential at $\rho = 0, \zeta = 0^+$ [cf. Eq. (12)]. The imaginary part of the Ernst potential in the disk depends on $\mu(1 - \rho^2/\rho_0^2)$ only, i.e.,

$$b = b_0(\mu[1 - \rho^2/\rho_0^2]), \qquad \zeta = 0^+. \tag{21}$$

Finally, the surface mass density σ defined by

$$\sigma = (1/2\pi)V_{,\zeta}|_{\zeta=0^+},$$

$$e^{2V} = e^{2U}[(1 + \Omega a)^2 - \Omega^2\rho^2 e^{-4U}] \tag{22}$$

is given by

$$\sigma = -\frac{\Omega}{2\pi\exp[V_0(\mu)]} \frac{b_0'(\mu[1 - \rho^2/\rho_0^2])}{\exp\{V_0(\mu[1 - \rho^2/\rho_0^2])\}}. \tag{23}$$

[Note that $b_0'(\mu) < 0$ for $\mu < \mu_0$.] In Fig. 1 we have plotted the radial variation of the proper surface mass density $\sigma_P = \sigma\exp(U - k)$ for several values of μ.

It should be noted that our exact results confirm the high accuracy of the approximate results obtained by Bardeen and Wagoner [3].

In conclusion, so far we have presented, in elementary functions and quadratures, the metric along the boundary of the space, i.e., on the axis of symmetry, in the disk, and at infinity. Although those data completely determine the interior solution, i.e., the mass density, the four-velocity and the gravitational field in the source (in the disk), and the multipole structure of the exterior solution, a more explicit representation of the vacuum region outside the axis would be desirable. The road there leads across a detailed analysis of the (linear) big integral equation, which is a standard reformulation of the Riemann-Hilbert problem solving our boundary value

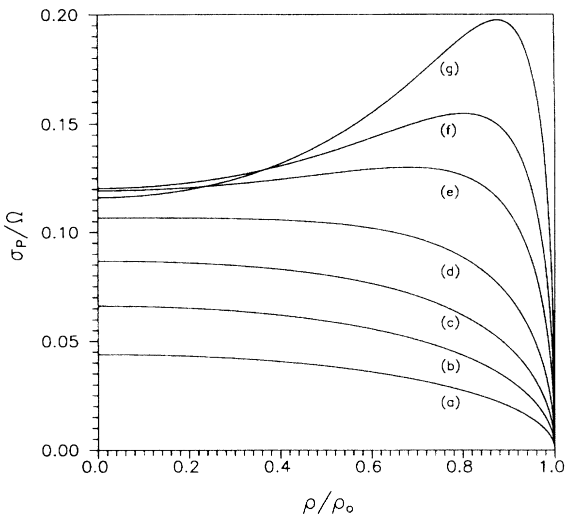

FIG. 1. Radial distribution of the proper surface mass density for parameter values $\mu = 0.1$ (a), $\mu = 0.25$ (b) , $\mu = 0.5$ (c), $\mu = 1.0$ (d), $\mu = 2.0$ (e), $\mu = 3.0$ (f), and $\mu = \mu_0 = 4.629\,66...$ (g). For small values of μ one recognizes the Newtonian profile $\propto (1 - \rho^2/\rho_0^2)^{1/2}$ of the Maclaurin disk. Note that $\rho_0 \rightarrow 0$ as $\mu \rightarrow \mu_0$.

problem [cf. [1], Eq. (27)]. There is a justified hope to make progress in that direction: It turns out that the solution procedure to the big integral equation is related to an unusual Bäcklund transformation of the Ernst equation.

Even if this correspondence should not prove useful, the big integral equation has a number of advantages to be emphasized: (i) It is a *linear* task which can be discretized without any difficulty. (ii) The coordinates enter the equation as parameters solely, such that the Ernst potential can *locally* be calculated with arbitrary accuracy.

The rigidly rotating dust disk is a first example for a nontrivial boundary value problem solvable by means of the inverse (scattering) method and likewise for a global solution describing a rotating isolated body.

The procedure might be applicable to more complicated rotating sources, e.g., to differentially rotating disks, to disks surrounded by rings (disconnected configurations), to a black hole surrounded by a disk, and to the exterior fields of rotating fluid balls.

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