Avalanches and $1/f$ Noise in Evolution and Growth Models

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We formally establish the relationship between spatial fractal behavior and long-range temporal correlations for a broad range of self-organized (and not self-organized) critical phenomena including directed percolation, interface depinning, and a simple evolution model. The recurrent activity at any particular site forms a fractal in time, with a power spectrum $S(f) \sim 1/f^{\tilde{d}}$. The exponent $\tilde{d} = (D - d)/z$, where d is the spatial dimension, D is the avalanche dimension, and z is the usual dynamical exponent. Theoretical results agree with numerical simulations.

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One of the most intriguing observations of natural phenomena is the widespread occurrence of both fractal scaling behavior [1] and $1/f$ type noise [2,3]. "Self-organized criticality" (SOC) [4] has been proposed as an explanation for these ubiquitous behaviors. In this picture, both are thought to be consequences of a dynamical process which drives large extended systems to an attractor that is poised at criticality. Thus far, though, this intuitively appealing connection has not been formally established. Existing evidence is primarily numerical [5].

Here, we demonstrate that fractal spatial and temporal behavior are intrinsically related for a broad range of critical phenomena including interface depinning [6] and growth [7], directed percolation (DP) [8], and the Bak-Sneppen (BS) evolution model [9]. By studying the *time* correlations in the *local* activity, we show that the temporal and spatial activity can be described as different cuts in the same underlying fractal. This fractal exhibits spatiotemporal complexity [10]. New scaling relations are derived, and many previously measured exponents are explained. Our most important results are as follows.

For all models considered, the activity at any particular site is recurrent in time; it is a "fractal renewal process" [11]. The scaling behavior of recurrent activity in time can be described by the fractal dimension $\tilde{d}$ of the return points on the one-dimensional time axis. This leads to an exponent $\tau_{\rm first} = \tilde{d} + 1$ for the first return times, or lifetimes, and to a $1/f^{\tilde{d}}$ power spectrum, where $0 \le \tilde{d} \le 1$. The exponent $\tilde{d}$ can be related to the dimension D of the avalanches through the relation $\tilde{d} = (D - d)/z$, where d is the spatial dimension and z is the usual dynamical exponent. We have performed numerical simulations to measure the recurrent activity for DP and the BS model. The results agree with our predictions based on known values of D and z of DP [12] and the BS model [13]. Our predicted exponents agree with measurements in one [14] and two [15] dimension for Sneppen's model of interface depinning [6]. For "parallel" interface models, where all unstable sites move at each time step [16,17], the exponent $\tilde{d}$ is identical to the usual exponent $\beta = \chi/z$ [7] characterizing the temporal development of the ensemble averaged width $w^2 = \langle (h - \langle h \rangle)^2 \rangle \sim t^{2\beta} \mathcal{F}(t/L^z)$. Measuring the lifetime exponents or power spectrum thus provides a new, *local* method to measure β.

Consider a one-dimensional time line for the activity at any particular site. The distribution of "hole" sizes, or intervals, separating subsequent return points of activity is given by the first return probability $P_{\rm first}(t)$, where t is the size of the hole. This distribution is normalizable; $\int P_{\rm first}(t) dt = 1$. The average total number of return points, $n(T)$, in an interval of length T is given by the fractal dimension of return points; i.e., $n(T) \sim T^{\tilde{d}}$. It can be related to the first return probability;

$$n(T) = T - n(T) \int_1^T P_{\rm first}(t) t \, dt , \qquad (1)$$

where $P_{\rm first}(t) \sim t^{-\tau_{\rm first}}$ for $t \gg 1$.If $\tau_{\rm first} \le 2$ then the divergence at the upper limit must cancel the T term, so that $T \sim n(T) T^{2-\tau_{\rm first}}$. This leads to the scaling relation

$$\tilde{d} = \tau_{\rm first} - 1 , \qquad (2)$$

which connects the fractal dimension of return points to the distribution of hole sizes. The quantity $P_{\rm all}(\mathbf{r}, t)$ is the probability that activity at position 0 at time 0 will be at $\mathbf{r}$ at time t. This quantity does not obey the same normalization condition as $P_{\rm first}$. Instead $\int P_{\rm all}(\mathbf{r}, t) d\mathbf{r} = N$, where N is the average number of active sites. Since $n(T)$ is simply the sum of all returns of activity to a particular site up to time T,

$$n(t + 1) - n(t) = P_{\rm all}(0, t) , \qquad (3)$$

where $P_{\rm all}(0, t) \sim t^{-\tau_{\rm all}}$ for $t \gg 1$. Equating these two expressions for $n(T)$ gives

$$\tau_{\rm first} + \tau_{\rm all} = 2 , \qquad (4)$$

which relates the "lifetime" exponents for the first and all returns of activity. Thus, the lifetime exponents are both determined by $\tilde{d}$. Since $P_{\rm all}(0, t)$ is the autocorrelation function, the power spectrum is simply [18]

$$S(f) = \int_{-\infty}^{+\infty} P_{\rm all}(0, t) \, e^{2\pi i f t} dt \sim \frac{1}{f^{\tilde{d}}} . \qquad (5)$$

For each model we consider, $\tilde{d}$ is derived by the studying geometry of avalanches. This establishes a formal connection between $1/f$ noise and fractal scaling behavior, i.e., spatiotemporal complexity, in these models. Model dependent behavior occurs between the upper critical and lower critical dimension. In the mean field limit, or above the upper critical dimension, the activity is barely able to return and $\tau_{\text{first}} = \tau_{\text{all}} = 1$. As a result, the power spectrum, $S(f) \sim 1/f^0$, corresponds to white noise. On the other hand, at the lower critical dimension, the activity becomes dense in time and $\tilde{d} \to 1$. In this case, the power spectrum $S(f) \sim 1/f$, with logarithmic corrections.

As the first example, we consider DP. In DP, a preferred direction, labeled by t, is chosen and bonds are oriented with respect to t. Percolation is only allowed in the direction of increasing t. Each bond exists with probability p. When $p = p_c$, the DP cluster can become infinitely large. Let us consider a large, finite cluster on a $(d + 1)$-dimensional lattice. A part of such a cluster is shown in Fig. 1 for $d = 1$. This cluster is asymmetric with respect to the t direction. Self-similarity requires that the time extent T scales with the spatial extent in any one of the d directions perpendicular to time, R, as $T \sim R^z$ where z is the usual dynamical exponent relating space and time. The total size of the cluster S scales with the spatial extent as $S \sim R^D$, where D is the avalanche dimension. In order to compute D, usually the cluster is partitioned into R^z equal time slices. Each such slice has $n_{\text{act}} \sim R^{d_f}$ parts of the cluster. Using this method of partitioning the cluster leads to $R^D \sim R^z R^{d_f}$, and the avalanche dimension $D = z + d_f$. We can also consider the cluster to be a composition of R^d one-dimensional fractals in time, each with $n(T)$ active sites [19]. Consequently,

$$R^D \sim R^d R^{z\tilde{d}}, \quad \tilde{d} = (D - d)/z, \tag{6}$$

and the power spectrum for DP, $S(f) \sim f^{-[1-(d-d_f)/z]}$. The lifetime exponents for DP are $\tau_{\text{first}} = 2 - (d - d_f)/z$ and $\tau_{\text{all}} = (d - d_f)/z$. We simulated bond DP on a square lattice in $1 + 1$ dimensions at $p = 0.6445$ for $L = 3000$. The data shown in Fig. 2, $\tau_{\text{first}} \simeq 1.86$ and $\tau_{\text{all}} \simeq 0.14$, are in agreement with the theoretical prediction $\tau_{\text{first}} \simeq 1.84$ and $\tau_{\text{all}} \simeq 0.16$ based on the exponents D and z in Ref. [12]. Also $S(f) \sim 1/f^{0.84}$ in $1 + 1$ dimensions.

We now consider the BS model of evolution [9], which is defined as follows: random numbers f_i are assigned independently to sites on a d-dimensional lattice. They are chosen from a uniform distribution between zero and one, $\mathcal{P}(f)$. At each step, the site with the lowest random number $f_{\min}$ is chosen. Then that site and its $2d$ nearest neighbors are assigned new random numbers which are also drawn from $\mathcal{P}$. After many updates have occurred, the system reaches a statistically stationary state in which the density of random numbers in the system with $f < f_c$ vanishes and is uniform above f_c. The activity pattern in the BS model, shown in Fig. 3, is a fractal in both space and time. In the steady state, the distribution of distances between sites of subsequent activity obeys a power law; i.e., $P(r) \sim r^{-\pi}$ where $\pi = 1 + D(2 - \tau)$ [20]. Here, the exponent τ describes the distribution of avalanche sizes, and D is the avalanche dimension. Since the BS model is sequential, and only one site is active at each time step, the time T is the same as the size S. As a result, the exponent z in Eq. (6) should be replaced with D. The lifetime exponents for the BS model are

$$\tilde{d} = 1 - d/D, \quad \tau_{\text{first}} = 2 - d/D, \quad \tau_{\text{all}} = d/D. \tag{7}$$

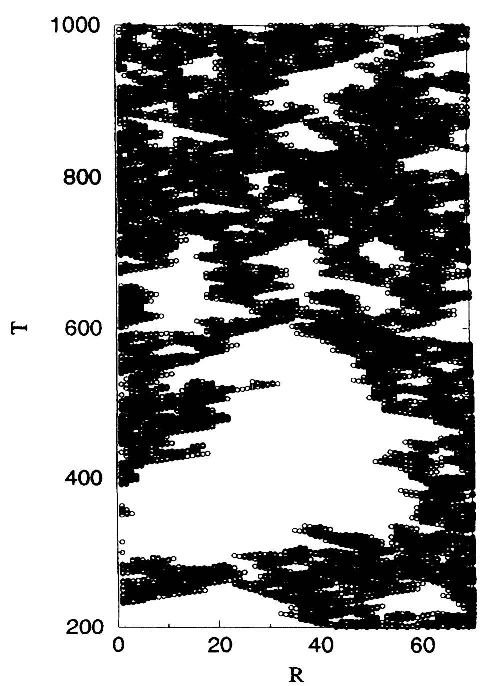

FIG. 1. Part of large directed percolation cluster in $d = 1$. The horizontal axis is a row of lattice sites and the vertical axis is parallel time. Note the appearance of holes of all sizes between returns to a given site.

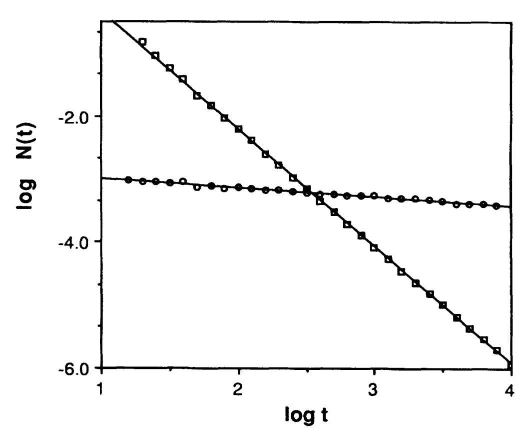

FIG. 2. Distribution of first return times (◇) and all return times (○) for directed percolation in $d = 1$. The measured exponents τ_{first} and τ_{all} are 1.86 and 0.14, respectively.

The power spectrum for the BS model is $S(f) \sim f^{-(1-d/D)}$.

It has been shown by Paczuski, Maslov, and Bak [13] that the BS model in the transient state approaching its SOC attractor can be mapped to an exactly equivalent BS branching process for $p < p_c$ away from the critical point. In addition, it was argued that the critical exponents for the BS model at the critical point, i.e., D and τ, are the same as DP. Given increased computational efficiency, we simulated the BS branching process directly, instead of the BS model. Inserting the values for DP [12,21] into Eq. (7) for the BS model gives ($d = 1$) $\tau_{\text{first}} \simeq 1.57$, $\tau_{\text{all}} \simeq 0.43$; ($d = 2$) $\tau_{\text{first}} \simeq 1.32$, $\tau_{\text{all}} \simeq 0.68$. These values are in agreement with values from our simulations of the BS branching process. In $d = 1$ we simulated the process at branching probability $p = 0.667$ and averaged over $\approx 10^9$ mutations to obtain Fig. 4. Our measured values are $\tau_{\text{first}} \simeq 1.58$ and $\tau_{\text{all}} \simeq 0.42$, quite close to the predicted values. Similar results were found for $d = 2$, at branching probability $p = 0.390$, $\tau_{\text{first}} \simeq 1.28$ and $\tau_{\text{all}} \simeq 0.70$. As seen in the figure, the value of τ_{first} for $d = 2$ is still increasing at the largest return times measured, so the true value of τ_{first} is likely to be larger. The predicted power spectrum is $S(f) \sim 1/f^{0.57}$ in $d = 1$ and $S(f) \sim 1/f^{0.32}$ in $d = 2$.

We now consider a simple lattice model that describes self-organized critical interface depinning, which was introduced by Sneppen [6]. An interface of size L^d defined on a discrete lattice $(\mathbf{x}, h)$ moves under the influence of random pinning forces $f(\mathbf{x}, h)$ assigned independently from $\mathcal{P}$. Initially, $h(\mathbf{x}) = 0$. Growth occurs by advancing the site on the interface with the smallest random pinning force by one step. Then a constraint is imposed for all nearest neighbor gradients, $|h(\mathbf{x}) - h(\mathbf{x}')| \leq 1$. This is met by advancing the heights of neighboring sites. This process is repeated indefinitely. When the total number of minimal sites chosen $s \sim L^D$, a system wide avalanche has pushed the interface to a critical depinned state. In the critical state, the width of the interface grows with system size as $w \sim L^\chi$, where χ is the roughness exponent. Consequently, the average volume of sites separating the critical interface configuration from the initial configuration scales as $L^{d+\chi}$, and

$$D = d + \chi\,. \tag{8}$$

as derived in Ref. [22]. If one covers the system wide avalanche with $N_{\text{col}}(S) \sim S^{d/D}$ time columns, the average number of returns $n(S) \sim S^{\bar{d}}$ obeys $S \sim N_{\text{col}}(S) n(S)$. Thus, Eq. (7) is valid for the Sneppen model with $D = d + \chi$. The results for the Sneppen model are

$$\tau_{\text{first}} = 2 - \frac{d}{d+\chi}\,, \quad \tau_{\text{all}} = \frac{d}{d+\chi}\,, \quad S(f) \sim f^{-\chi/(d+\chi)}. \tag{9}$$

As Tang and Leschhorn showed [23,24] by mapping the one-dimensional problem to "rotated" DP the roughness

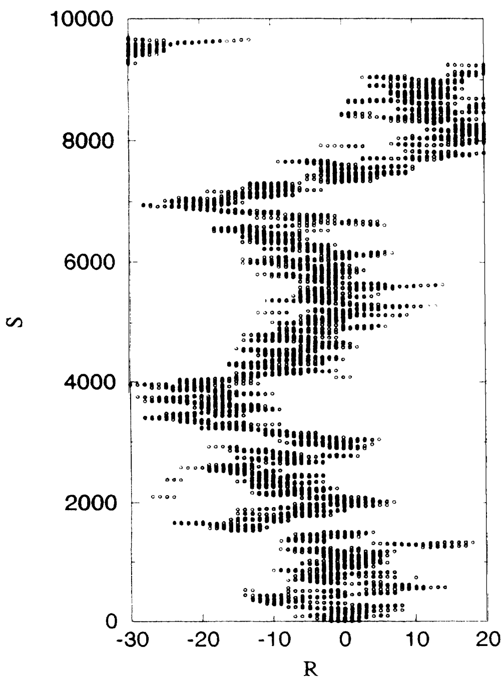

FIG. 3. Cluster of activity in the $d = 1$ Bak-Sneppen evolution model. The horizontal axis is a row of lattice sites and the vertical axis is sequential time. Note the appearance of holes of all sizes between returns to a given site.

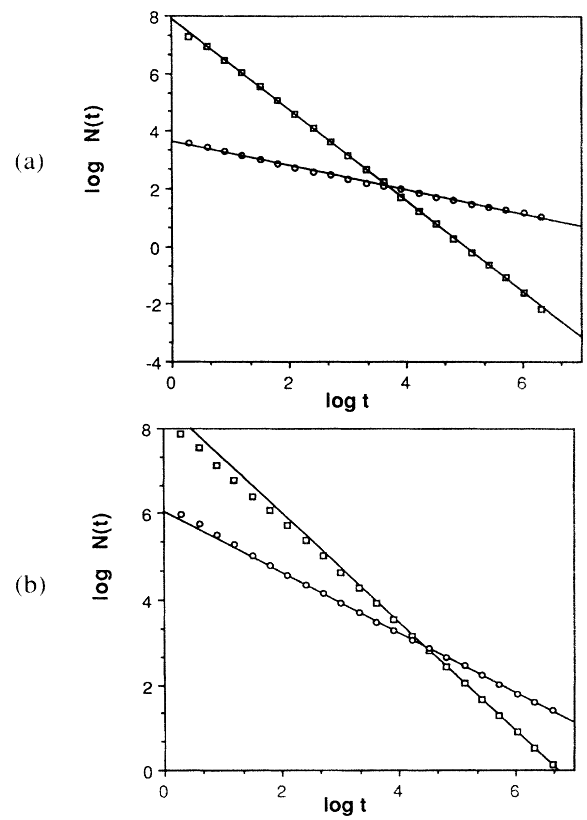

FIG. 4. Distribution of first return times (◇) and all return times (◯) for the Bak-Sneppen evolution model. (a) $d = 1$; (b) $d = 2$. In $d = 1$, $\tau_{\text{first}} \simeq 1.57$, and $\tau_{\text{all}} \simeq 0.43$. In $d = 2$, $\tau_{\text{first}} \simeq 1.26$, and $\tau_{\text{all}} \simeq 0.43$.

exponent $\chi \simeq 0.63$ in $d = 1$. Simulations give $\chi \simeq 0.5$ in $d = 2$ [15]. Using these values for χ in Eq. (9) we obtain $\tau_{\rm all} \simeq 0.61$ and $\tau_{\rm first} \simeq 1.39$ in $d = 1$ to be compared with $\tau_{\rm all} = 0.62 \pm .04$, $\tau_{\rm first} = 1.35 \pm 0.1$ measured by Sneppen and Jensen [14]. In $d = 2$, we predict $\tau_{\rm all} \simeq 0.8$ and $\tau_{\rm first} \simeq 1.2$ compared with the measured values $\tau_{\rm all} \simeq 0.8 \pm 0.01$ and $\tau_{\rm first} \simeq 1.2 \pm 0.1$ [15]. The power spectrum from Eq. (9) is $S(f) \sim f^{-0.39}$ in $d = 1$ and $S(f) \sim f^{-0.20}$ in $d = 2$, based on the measured value of χ.

Our results also apply to a similar model for depinning [16,17] that does not exhibit SOC. Instead of advancing the site with the lowest random pinning force, all unstable sites with $f < F$ are advanced in parallel. Then the system is relaxed to meet the gradient constraint. For F slightly below f_c the interface relaxes by large avalanches each time F is incremented until the interface gets stuck again at this new higher value of F. Because of the parallel rules of dynamics, the time T differs from the size S, and the dynamical exponent z must be included as an independent exponent. Substituting Eq. (8) into (6) gives $\tilde{d} = \chi/z$ for the parallel model. This is just the usual exponent β that characterizes the temporal development of the width, $w \sim t^\beta F(t/L^z)$. Thus

$$\tau_{\rm first} = 1 + \beta, \quad \tau_{\rm all} = 1 - \beta, \quad S(f) \sim 1/f^\beta. \quad (10)$$

In $d = 1$, the exponent β has been measured to be $\simeq 0.63$ [16] and 0.68 ± 0.04 [17] by examining the interface width. It is important to note that making use of Eq. (10) and measuring the lifetime exponents provides a new, local, method to measure β.

Equation (9) is expected to describe depinning for any "reasonable" sequential interface model in a quenched random medium, where the most unstable site moves at each step. Also, Eq. (10) applies for parallel models where all unstable sites move in parallel at each time step. We have also studied a "linear" interface model, which before is believed to describe the depinning of an interface in the random field Ising model [25]. We have measured the avalanche dimension D for a discretized self-organized version, where the force at each site is the same as that used by Leschhorn in Ref. [26], although we advance only the most unstable site. In $d = 1$, $D = 2.23 \pm 0.01$, and in $d = 2$, $D \simeq 2.725 \pm 0.01$. Substituting these measured values in Eq. (8) gives $\chi \simeq 1.23$ in $d = 1$ and $\chi \simeq 0.73$ in $d = 2$. These values for χ are in agreement with numerical simulations by Leschhorn [26], but higher than the prediction $\chi = (4 - d)/3$ from functional renormalization group calculations [27].

The "game of life" is a two-dimensional cellular automaton which has been found to be at or very near criticality [28]. In addition, it has been conjectured [13] that the game of life belongs to the same universality class as DP. Numerical measurements of the avalanche distribution exponents [29] support this conjecture. It would be interesting to also measure the lifetime exponents, which are predicted to be $\tau_{\rm first} \simeq 1.54$ and $\tau_{\rm all} \simeq 0.46$, and $S(f) \sim 1/f^{0.54}$.

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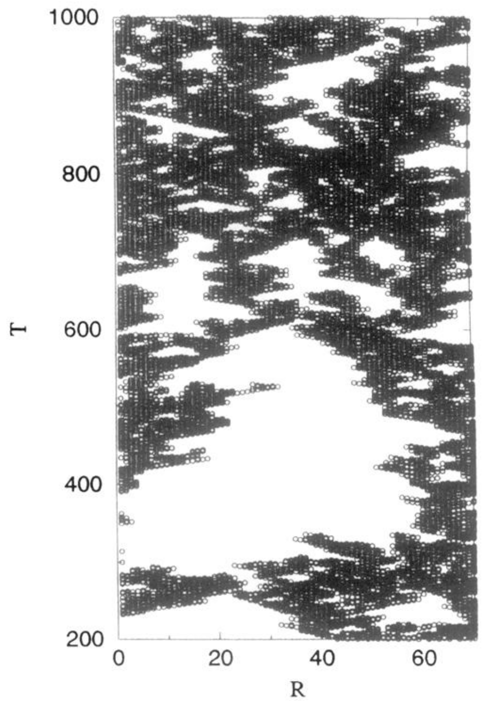

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