Statistical Mechanics for a Class of Quantum Statistics

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Generalized statistical distributions for identical particles are introduced for the case where filling a single-particle quantum state by particles depends on filling states of different momenta. The system of one-dimensional bosons with a two-body potential that can be solved by means of the thermodynamic Bethe ansatz is shown to be equivalent thermodynamically to a system of free particles obeying statistical distributions of the above class. The quantum statistics arising in this way are completely determined by the two-particle scattering phases of the corresponding interacting systems. An equation determining the statistical distributions for these statistics is derived.

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There are several approaches applicable to the investigation of generalized statistics for identical particles in one dimension [1–6]. In the Schrödinger picture, statistics are defined by conditions which many-particle wave functions must satisfy at the boundary of the configuration space for identical particles (for N identical particles on a line, the latter is $\mathbf{R}^N/S_N$ and can be represented by the region $x_1 \geq x_2 \geq \cdots \geq x_N$ in $\mathbf{R}^N$) [1,3,4]. The Heisenberg quantization of identical particles [2] deals with the algebra of observables for a system of identical particles. Haldane [5] has proposed a definition of fractional statistics in which "statistical" interaction is characterized by the change of the number of single-particle states, by $-g$, in adding one particle into the system ($g = 0$ and $g = 1$ correspond to bosons and fermions). This definition was applied to excitations in the fractional quantum Hall effect (see also [7]).

A family of statistics (which was originally introduced in [2] for two particles in terms of the Heisenberg quantization of identical particles) was defined, in the Schrödinger picture, by the boundary conditions on the wave functions $\psi \sim \prod_{i<j}(x_i - x_j)^\alpha$ as $x_i - x_j \to +0$, with $\alpha \geq 0$, where $x_1, x_2, \ldots,$ are particle coordinates (with $\alpha = 0$ and $\alpha = 1$ for bosons and fermions) [3]. We call these statistics *1D fractional statistics*. The relation to anyons in two spatial dimensions [8] was established: 1D fractional statistics of order α is relevant to anyons with statistical parameter $\theta = \pi\alpha$ in the lowest Landau level where the dynamics of the system becomes effectively one dimensional [9]. An equivalence of a system of free particles obeying 1D fractional statistics of order α to a system of one-dimensional bosons interacting through the two-body inverse square potential $V(x) = \alpha(\alpha - 1)/x^2$ (Calogero-Sutherland system [10]) was also stated [2,3].

Another family of statistics was introduced in Ref. [1] for which two-particle wave functions for the relative motion obey $(\partial/\partial x)\psi(x) \to c\psi(x)$ as the relative coordinate $x \to +0$, with $0 \leq c < \infty$ ($c = 0$ and $c \to \infty$ correspond to Bose and Fermi statistics). The generalization of the N-particle wave function was given in Ref. [4]. These statistics can also be modeled by means of an interaction: a free system with wave functions satisfying the above conditions is equivalent to a system of bosons with the δ-function interaction potential $V(x) = c\delta(x)$ [2,4].

In Ref. [11] an approach to generalization of statistics in the framework of statistical mechanics was suggested by the author where statistical distributions for identical particles more general than the Bose and Fermi distributions were introduced. It was assumed in [11] that single-particle quantum states are filled by particles independent of each other (as for Bose and Fermi statistics); in other words, "statistical" interaction occurs only between particles in the same state. No assumptions were made concerning allowed occupation numbers. In [6] statistical distribution for the 1D fractional statistics was derived which turned out to belong to the class of distributions introduced in [11].

In this paper we introduce statistical distributions for identical particles for a more general case where statistical interaction also exists between particles of distinct momenta. We next show that integrable one-dimensional bosonic systems are equivalent to free systems with particles obeying statistical distributions of this class. New statistics for identical particles are generated in this way. The statistics modeled by the δ-function interaction [1] belong to this class.

We start from the expression for the grand partition function of a quasiclosed subsystem resulting from the Gibbs distribution [12]:

$$\Xi^{\rm subsyst} = \sum_{N,n} \exp[\beta(\mu N - E_{N,n})], \tag{1}$$

where $\beta = 1/T$ is the inverse temperature, μ is the chemical potential, the index n numbers states of the subsystem at given particle number N in it, and $E_{N,n}$ are the energies of these states.

Consider a gas of free identical particles occupying an interval of length L on a line [13]. If single-particle quantum states are filled independently of each other, one can choose particles in the same state as a quasiclosed subsystem [11,12]. Here we consider the case where filling a state depends on filling all the other states, that is, there is a statistical interaction between particles of different momenta. Therefore we apply the formula (1) to the whole system.

The total number and energy of free particles are $N = \sum_k N_k$ and $E = \sum_k \epsilon_k N_k$, where N_k is the number of particles of momentum k and ϵ_k is the energy of a particle of momentum k [14]. Then (1) reads

$$\Xi = \sum_{\{N_k\}} \prod_k (x_k)^{N_k}, \qquad (2)$$

where

$$x_k = \exp[\beta(\mu - \epsilon_k)] \qquad (3)$$

is the Gibbs factor for a single-particle state of momentum k, $\{N_k\}$ denotes the set of the numbers N_k with all possible momenta for a given distribution of particles over momenta, and the summation in (2) goes over all allowed sets $\{N_k\}$ which characterize statistics of particles.

If states of different momenta are statistically independent [15], the partition function (2) is factored into the product of the partition functions corresponding to separate states: $\Xi = \prod_k \Xi_k$. The latter decomposition does not hold for the case under consideration.

To treat the general statistical distributions, we will not make any assumptions concerning allowed sets of the occupation numbers $\{N_k\}$. Instead we observe that according to (2) the function Ξ depends on μ and T only through the combinations x_k of (3) with all possible momenta. Therefore the dependence of Ξ on μ and T can be written down explicitly as

$$\Xi = \Xi(\{x_k\}), \qquad (4)$$

where the notation $\{x_k\}$ is similar to $\{N_k\}$. We will regard this dependence as arbitrary except for the Boltzmann limit condition (see below).

The average particle number is given by the thermodynamic identity $\bar{N} = -(\partial\Omega/\partial\mu)_T$, where $\Omega = -\beta^{-1}\ln\Xi$. With Ξ of (4), we get

$$\bar{N} = \sum_k x_k \frac{\partial}{\partial x_k} \ln \Xi(\{x_{k'}\}). \qquad (5)$$

We also introduce the equilibrium distribution of particles over momenta n_k. On the basis of (5) we identify

$$n_k = x_k \frac{\partial}{\partial x_k} \ln \Xi(\{x_{k'}\}), \qquad (6)$$

so that $\sum_k n_k = \bar{N}$. The statistical distribution [or, equivalently, the partition function (4)] determines all the thermodynamic quantities for a free gas. For example, the pressure of the gas is obtained from the relation $\Omega = -PL$ to give

$$P = (\beta L)^{-1} \ln \Xi(\{x_k\}). \qquad (7)$$

The statistical distribution (6) has to satisfy the Boltzmann limit condition, that is, it has to go to the Boltzmann distribution $n_k^{\mathrm{Boltzmann}} = x_k$ as all $x_{k'}$'s vanish (when the average occupation numbers of quantum states are small and effects of the gas degeneracy are negligible [12]). This gives the constraints

$$(\partial/\partial x_k) \ln \Xi(\{x_{k'}\}) \to 1 \qquad (8)$$

as all $x_{k'}$ vanish.

Consider now the statistics for which the partition function (4) admits an expansion in integer powers of the Gibbs factors $\{x_k\}$. It is more convenient to treat $\ln\Xi$ instead of Ξ:

$$\ln \Xi = \sum_k x_k + \frac{1}{2}\sum_{k,k'} a_{kk'} x_k x_{k'} + \cdots. \qquad (9)$$

That $\ln\Xi$ goes to zero as all x_k's vanish means that the vacuum state is supposed to be nondegenerate. The coefficients of x_k's in (9) were chosen as equal to one in order to satisfy the conditions of the existence of the Boltzmann limit (8). The coefficients $a_{kk'}$ are symmetric, by definition, under permutations of their subscripts. The statistical distribution (6) corresponding to (9) is

$$n_k = x_k \left(1 + \sum_{k'} a_{kk'} x_{k'} + \cdots\right). \qquad (10)$$

For the distributions discussed in [11] one has $a_{kk'} \propto \delta_{kk'}$ (in particular, bosons and fermions have $a^B_{kk'} = \delta_{kk'}$ and $a^F_{kk'} = -\delta_{kk'}$). $a_{kk'} \neq 0$ for $k \neq k'$ corresponds to the occurrence of statistical interaction between particles of momenta k and k'.

To the expansions (9) and (10) there corresponds the virial expansion for a gas of free particles. Equation (9) can be rewritten as

$$-\beta\Omega = \ln\Xi = b_1 z + b_2 z^2 + \cdots, \qquad (11)$$

where $z = e^{\beta\mu}$ is the fugacity, $b_1 = \sum_k \exp(-\beta\epsilon_k)$, and

$$b_2 = \frac{1}{2}\sum_{k,k'} a_{kk'} \exp[-\beta(\epsilon_k + \epsilon_{k'})].$$

It is seen from (11) that $b_1, b_2, \ldots,$ are nothing but the cluster coefficients [16] which determine the coefficients of the virial expansion ($\rho = N/L$ is the gas density) $P\beta = \rho + A_2\rho^2 + \cdots$. In particular, $A_2 = -b_2 L/b_1^2$ [16]. With the above expressions for b_1 and b_2, the latter formula evaluates the second virial coefficient carrying information on the statistics and dispersion law of particles.

Consider now an integrable one-dimensional system, namely a system of interacting bosons on a ring of circumference L with periodic boundary conditions which is governed by the Hamiltonian

$$H = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} V(x_i - x_j). \qquad (12)$$

We assume that the system can be solved by means of the TBA [15,17]. The TBA was introduced in [15] to treat the thermodynamics of the systems described by the Bethe ansatz form wave functions. It was then observed that for applicability of the TBA it is sufficient that the wave functions have the Bethe ansatz form only in the asymptotic regions where particles are far apart, which enables one to consider systems with long-term interactions [17].

In [6] we used the TBA to show that a system of bosons interacting through an inverse square potential is equivalent thermodynamically to a free system. Here we generalize that statement to the potential $V(x_i - x_j)$ of the general form.

We first observe that the applicability of the TBA implies the additivity of the energy of the system (12):

$$E = \int \frac{1}{2} k^2 \rho(k) L \, dk \,, \qquad (13)$$

where $\rho(k)$ is the distribution function of the (interacting) particles in the TBA scheme. The expression (13) can be interpreted as the energy of a system of free particles with energies $\epsilon_k = \frac{1}{2}k^2$ if we identify

$$2\pi\rho(k) = n_k \,, \qquad (14)$$

where n_k is the statistical distribution of particles in the equivalent free system.

The other thermodynamic quantities of the interacting boson system can be found from the expression for the pressure [15,17]

$$P = \beta^{-1} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln(1 + e^{-\beta\tilde{\epsilon}(k)}) \,, \qquad (15)$$

where the so-called pseudoenergy $\tilde{\epsilon}(k)$ being also a function of μ and T is determined by the TBA equation

$$\tilde{\epsilon}(k) = -\mu + \frac{1}{2}k^2 + \beta^{-1} \times \int_{-\infty}^{\infty} \frac{L\,dk'}{2\pi} \varphi(k - k') \ln(1 + e^{-\beta\tilde{\epsilon}(k')}) \,, \qquad (16)$$

with $\varphi(k - k')$ the derivative of the phase shift $\theta(k - k')$ for the scattering of two particles of momenta k and k' in the potential $V(x)$: $\varphi(k - k') = (1/L)\partial\theta(k - k')/\partial k$.

Now compare (15) with the expression for the pressure of a gas of free particles (7). This gives the identification

$$\ln \Xi = \sum_k \ln \Lambda_k \,, \qquad (17)$$

where $\Lambda_k = 1 + \exp[-\beta\tilde{\epsilon}(k)]$. From (16), the equation for the Λ_k is obtained:

$$\Lambda_k = 1 + x_k \prod_{k'} (\Lambda_{k'})^{-\varphi_{k-k'}} \,, \qquad (18)$$

where x_k is given by (3) with the energy of free particles $\epsilon_k = \frac{1}{2}k^2$ [in accordance with (13) and (14)]. This equation shows that Λ_k and, according to (17), the partition function of the equivalent free system Ξ depend on μ and T only via the combinations x_k, that is, Ξ fits exactly into the form (4). Expanding Λ_k in powers of x_k's, we obtain from (18)

$$\Lambda_k = 1 + x_k \left[1 - \sum_{k'} \varphi_{k-k'} x_{k'} + \cdots \right].$$

Inserting this into (17), we see that the corresponding statistical distribution n_k (6) falls into the class of distributions of the form (10), with

$$a_{kk'} = -\varphi_{k-k'} - \frac{1}{2}\delta_{k-k'} \,. \qquad (19)$$

Thus the integrable boson system in question may indeed be interpreted as equivalent to a system of free particles obeying statistics characterized by the expansions (9) and (10).

This equivalence can also be extended to the nonequilibrium case. The nonequilibrium entropy of the integrable system at hand in the thermodynamic limit reads

$$S = \int_{-\infty}^{\infty} [(\rho + \rho_h)\ln(\rho + \rho_h) - \rho \ln \rho - \rho_h \ln \rho_h] L \, dk \,,$$

where ρ_h is the density of the holes arising in the TBA scheme. Eliminating ρ_h from this expression with the aid of the relation [15,17]

$$2\pi(\rho + \rho_h) = 1 - \int_{-\infty}^{\infty} \varphi(k - k')\rho(k') L \, dk'$$

and using, in addition, (14), we get

$$S = \sum_k \left\{ \left[1 - \sum_{k'} \varphi_{k-k'} n_{k'} \right] \ln \left[1 - \sum_{k'} \varphi_{k-k'} n_{k'} \right] - n_k \ln n_k - \left[1 - n_k - \sum_{k'} \varphi_{k-k'} n_{k'} \right] \ln \left[1 - n_k - \sum_{k'} \varphi_{k-k'} n_{k'} \right] \right\}. \qquad (20)$$

This expression can be interpreted as the expression for the nonequilibrium entropy of the equivalent free system. Indeed, varying (20) with respect to n_k, at the particle number $N = \sum_k n_k$ and the total energy $E = \sum_k \epsilon_k n_k$ held constant, with the associated Lagrange multipliers $\beta\mu$ and $-\beta$, gives as an extremum condition the relation for the equilibrium statistical distribution $\delta S/\delta n_k + \ln x_k = 0$ or

$$n_k \prod_{k_1} \left[1 - \sum_{k'} \varphi_{k_1-k'} n_{k'} \right]^{\varphi_{k-k_1}} \left[1 - n_{k_1} - \sum_{k'} \varphi_{k_1-k'} n_{k'} \right]^{-\delta_{k,k_1} - \varphi_{k-k_1}} = x_k \,, \qquad (21)$$

with first terms of the expansion of n_k from (21) in powers of x_k's coinciding with (19).

For a system of bosons interacting with the inverse square potential, $\varphi_{k-k'} = (\alpha - 1)/\delta_{kk'}$ [17]. In this case Eq. (21) reduces to the equation for the statistical distribution for 1D fractional statistics [6]:

$$n_k(1 - \alpha n_k)^{-\alpha}[1 + (1 - \alpha)n_k]^{\alpha - 1} = x_k\,. \qquad (22)$$

That $\varphi_{k-k'} \propto \delta_{kk'}$ in this case corresponds to statistical interaction only between particles of the same momenta. Contrary to that, for bosons with the δ-function interaction, $\varphi(k - k') = -2c/[c^2 + (k - k')^2]$ [15], which implies statistical interactions between particles of distinct momenta. The function $\varphi(k - k')$ may be regarded as a measure of statistical interaction between particles of momenta k and k'.

The expression for the nonequilibrium entropy (20) can serve as a starting point for establishing correspondence with Haldane's description of statistical interaction [5]. Applying Haldane's definition of fractional statistics to particles of the same momentum and using the Boltzmann formula $S = \sum_k \ln \Gamma_k$, where Γ_k is the statistical weight of particles of momentum k (the dimension of the space of many-particle states for particles of momentum k in Haldane's terminology), we come to (20) with $\varphi_{k-k'} = (g - 1)\delta_{kk'}$, corresponding to 1D fractional statistics of order g (for $g \geq 0$) [18]. Similarly, the expression (20) with generic $\varphi_{k-k'}$ enables one to formulate a definition in the spirit of Haldane for the statistics discussed in this paper [19].

In conclusion, we have introduced statistical distributions (6) and (10) for ideal quantum gases with statistical interaction between particles of different momenta. One-dimensional integrable bosonic systems were interpreted as equivalent to free systems of this class. The relevant statistical distributions are determined by Eq. (21) with different $\varphi_{k-k'}$ corresponding to different integrable systems. Note that statistical distributions, originally obtained from Eq. (21) for a quadratic dispersion law of particles, can then be used for evaluation of thermodynamic quantities of (quasi)particles having different (e.g., linear) dispersion laws.

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Note added.—After this paper had been submitted, a closely related report [20] appeared also stating the above equivalence of one-dimensional integrable systems to free systems, starting from a description of statistical interactions in the spirit of Haldane. In another report [21], a Haldane-like approach was used to derive the statistical distribution for 1D fractional statistics (22). I am grateful to J. M. Leinaas who supplied me with these reports.

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