Stability of Solid Propellant Combustion

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The exact analytical solution of the stability problem of combustion in a solid propellant is obtained. The problem is solved for the complete system of equations and for a reaction zone of finite thickness.

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The stability of a combustion wave is one of the most important problems in combustion theory. The solution of the problem can be essentially simplified if the characteristic scales of the processes governing the combustion are small in comparison with the characteristic dimensions of the problem. A common approach to the stability problem of a flame front is to treat the flame front or some inner zone (usually it is the chemical reaction zone) as a surface of discontinuity of zero thickness (Refs. [1,2], and references therein). The model with the discontinuity is widely used to solve the flame stability problem of both gaseous [3,4] and condensed fuels [5–8]. However, the stability problem thus formulated with a surface of discontinuity leads to another difficulty, since it requires an additional boundary condition (jump condition at the surface of discontinuity), which in principle cannot be obtained or justified within the scope of the discontinuity model [9,10]. Because of this, the boundary condition at the discontinuity is introduced as a "physically reasonable" boundary condition. The sensitivity of the solution obtained in this way and the accuracy of the assumed boundary condition can be verified only by solving the complete problem with finite thickness and real structure of the fronts [10]. Such an examination can be a rather difficult mathematical problem. For these reasons and because the flame is a typical example of a wide class of the reacting waves in the deflagration regime (ionizing wave, phase transition wave, ablation wave, etc.), the exact solution of the combustion stability problem is of special importance.

In this paper we demonstrate the exact analytical solution for the problem of the stability of gasless combustion propagating in solid propellant. An example of such a flame is the combustion of thermites [11]. Since the fuel and the combustion products are in a solid phase, there is no diffusion and the combustion propagates by means of thermal conduction only. It was found [1,2] that combustion in a solid propellant is unstable and the fastest (one dimensional) instability results in a pulsating regime of the combustion front. For the case of large activation energy $E \gg T_2^2/(T_2 - T_1)$, where T_1 and T_2 are the temperature of the fuel and the combustion products respectively, the chemical reaction zone is thin in comparison with the preheat zone. The solution of the stability problem was obtained in [5–8] by regarding the reaction zone as the inner discontinuity surface and assuming that the temperature perturbation is continuous at the discontinuity surface. The growth rate for the fastest perturbations was found to be

$$\sigma = \frac{u_f^2}{\chi} \frac{E^2(T_2 - T_1)^2}{16T_2^4}, \tag{1}$$

where u_f is the velocity of the combustion wave and χ is the thermal diffusivity.

It is easily seen that there is a contradiction between this solution and the basic assumption, since the temperature perturbations change on the length scale $\approx \sqrt{\chi/\sigma}$, which is of the same order as the thickness of the reaction zone $\delta_{\text{ch}} \approx \chi T_2/u_f E$. Thus the perturbations are localized inside the chemical reaction zone, which therefore cannot be treated as a surface of zero thickness.

Let us consider the spectral problem of the flame stability in condensed matter, taking into account the finite thickness of the reaction zone. The diffusion is negligible and propagation of the flame is described by the equations

$$\frac{\partial T}{\partial t} + \mathbf{u}_f \nabla T = \nabla(\chi \nabla T) + \frac{aQ}{c\tau_r} \exp(-E/T), \tag{2}$$

$$\frac{\partial a}{\partial t} + \mathbf{u}_f \nabla a = -\frac{a}{\tau_r} \exp(-E/T), \tag{3}$$

where a is the fuel concentration, and Q is the heat release of the reaction. We consider a first order chemical reaction with the Arrhenius reaction rate; τ_r is a constant with units of time; c is the specific heat. The steady planar flame propagates along the z axis.

It is convenient to introduce the following dimensionless variables and parameters

$$\Theta = T/T_2, \quad \alpha = aQ/cT_2, \quad \kappa = \chi/\chi_2, \tag{4a}$$

$$\xi = zu_f/\chi_2, \quad \tau = tu_f^2/\chi_2, \quad \mathcal{E} = E/T_2, \quad \Lambda = \frac{\chi_2}{u_f^2 \tau_r}. \tag{4b}$$

The velocity of the steady flame is determined by the eigenvalue Λ of the following system

$$\frac{d}{d\xi}\left(\kappa \frac{d\Theta}{d\xi}\right) - \frac{d\Theta}{d\xi} + \alpha \Lambda \exp(-\mathcal{E}/\Theta) = 0, \tag{5}$$

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$$\frac{d\alpha}{d\xi} + \alpha\Lambda \exp(-\mathcal{E}/\Theta) = 0, \tag{6}$$

with the boundary conditions

$$\Theta = \Theta_1, \quad \alpha = \alpha_1 \quad \text{for } \xi \to -\infty, \tag{7}$$

$$\frac{d\Theta}{d\xi} = 0, \ \alpha = 0 \quad \text{for } \xi \to \infty. \tag{8}$$

The solution of the problem (5)–(8) exists as an intermediate asymptotic in the limit $\Lambda \exp(-\mathcal{E}/\Theta_1) \approx 0$, which means the reaction time of the fuel in the initial state is large in comparison with other time scales of the problem [1].

In the limit $\mathcal{E}(1 - \Theta_1) \gg 1$ the fuel concentration is constant and equal to the initial value α_1 everywhere up to the thin chemical reaction zone of thickness $1/\mathcal{E}$. For the steady flame the temperature changes on a length scale of order unity in dimensionless variables, while inside the reaction zone the temperature changes slowly as $1/\mathcal{E}$. Thus, inside the reaction zone Eqs. (5) and (6) to within terms of order $1/\mathcal{E}$ can be written as

$$\frac{d\Theta}{d\xi} = \alpha, \tag{9}$$

$$\frac{d^2\Theta}{d\xi^2} + \frac{d\Theta}{d\xi}\Lambda \exp(-\mathcal{E}) \exp[\mathcal{E}(\Theta - 1)] = 0, \tag{10}$$

with the boundary conditions $d\Theta/d\xi = 1 - \Theta_1$ for $\mathcal{E}(1 - \Theta) \gg 1$ and $d\Theta/d\xi = 0$ for $\Theta = 1$. The solution of Eq. (10) is

$$\mathcal{E}(\Theta - 1) - \ln\{1 - \exp[\mathcal{E}(\Theta - 1)]\} = \mathcal{E}(1 - \Theta_1)\xi. \tag{11}$$

The corresponding eigenvalue is

$$\Lambda = \mathcal{E}(1 - \Theta_1)\exp(\mathcal{E}). \tag{12}$$

For planar unperturbed flow, the small perturbations can be chosen in the form

$$\tilde{\varphi}(\xi, \zeta, \tau) = \tilde{\varphi}(\xi)\exp(S\tau + iK\zeta), \tag{13}$$

where $\zeta = xu_f/\chi_2$, $S = \sigma\chi_2/u_f^2$, and $K = k\chi_2/u_f$ are the dimensionless coordinate, instability growth rate, and perturbation wave number, respectively.

The linearized equations for the small perturbations follow from the system (2), (3)

$$\frac{d^2}{d\xi^2}(\kappa\tilde{\Theta}) - \frac{d\tilde{\Theta}}{d\xi} - (S + \kappa K^2)\tilde{\Theta} + \Lambda(\tilde{\alpha} + \alpha\mathcal{E}\tilde{\Theta}/\Theta^2)\exp(-\mathcal{E}/\Theta) = 0, \tag{14}$$

$$\frac{d\tilde{\alpha}}{d\xi} + S\tilde{\alpha} + \Lambda(\tilde{\alpha} + \alpha\mathcal{E}\tilde{\Theta}/\Theta^2)\exp(-\mathcal{E}/\Theta) = 0. \tag{15}$$

The boundary conditions for Eqs. (14) and (15) are that the perturbations vanish as $\xi \to \pm\infty$.

Taking into account that $\Theta \approx 1$ inside the reaction zone we obtain from (15)

$$\tilde{\alpha} = \mathcal{E}\alpha \exp(-S\xi)\int_{-\infty}^{\xi} \tilde{\Theta}\frac{d\ln(\alpha)}{d\eta}\exp(S\eta)d\eta. \tag{16}$$

Assuming that $S \sim \mathcal{E}^2$, and perturbations of the temperature $\tilde{\Theta}$ change on the length scale $S^{-1/2} \sim \mathcal{E}^{-1}$, which is comparable to the thickness of the reaction zone, we obtain from (16) to within terms of order $\mathcal{E}^{-1}$

$$\tilde{\alpha} = \frac{\mathcal{E}}{S}\frac{d\alpha}{d\xi}\tilde{\Theta}. \tag{17}$$

Substituting (17) into (14) and taking into account the zero-order terms of the expansion in $\mathcal{E}^{-1} \ll 1$, we obtain the equation for the temperature perturbation,

$$\frac{d^2\tilde{\Theta}}{d\xi^2} - \tilde{\Theta}(S + K^2 + \mathcal{E}d\alpha/d\xi) = 0. \tag{18}$$

Taking into account (9) and the solution (11) for the structure of the reaction zone we have

$$\frac{d\alpha}{d\xi} = -\frac{1}{4}\mathcal{E}(1 - \Theta_1)^2\cosh^{-2}[\mathcal{E}(1 - \Theta_1)\xi/2]. \tag{19}$$

Substituting (19) into (18) and introducing the variables

$$\eta = \mathcal{E}(1 - \Theta_1)\xi/2, \quad S_\eta = 4\frac{S + K^2}{\mathcal{E}^2(1 - \Theta_1)^2}, \tag{20}$$

we obtain

$$\frac{d^2\tilde{\Theta}}{d\eta^2} - (S_\eta - \cosh^{-2}\eta)\tilde{\Theta} = 0, \tag{21}$$

which is the Schrödinger equation for the one-dimensional motion of a particle of unit mass in the potential well $\cosh^{-2}\eta$. Equation (21) has an exact solution [12] with the following eigenvalues

$$S_{\eta,n} = \tfrac{1}{4}(-1 - 2n + \sqrt{5}), \tag{22}$$

where $n = 0, 1, 2, \ldots$. Only the mode $n = 0$ corresponds to growing perturbations ($S > 0$), so the instability growth rate is

$$S = \frac{\sqrt{5} - 1}{16}\mathcal{E}^2(1 - \Theta_1)^2 - K^2. \tag{23}$$

In the dimensional variables the instability growth rate is

$$\sigma = \frac{\sqrt{5} - 1}{16}\frac{u_f^2}{\chi_2}\frac{E^2(T_2 - T_1)^2}{T_2^4} - \chi_2 k^2. \tag{24}$$

The exact solution for the instability growth rate differs from the result (1) by the factor $\sqrt{5} - 1$. Thus even for the limiting case of large activation energy, solution in the discontinuity model differs from the exact solution. It is very likely that for the case of finite activation energy the difference between the exact solution and the solution obtained in the discontinuity model will be much more pronounced.

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