Obstructions to Shadowing When a Lyapunov Exponent Fluctuates about Zero

Silvina Dawson
Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

Celso Grebogi
Department of Mathematics, University of Maryland, College Park, Maryland 20742

Tim Sauer
Department of Mathematical Sciences, George Mason University, Fairfax, Virginia 22030

James A. Yorke
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742
(Received 17 December 1993)

We study the existence or nonexistence of true trajectories of chaotic dynamical systems that lie close to computer-generated trajectories. The nonexistence of such shadowing trajectories is caused by finite-time Lyapunov exponents of the system fluctuating about zero. A dynamical mechanism of the unshadowability is explained through a theoretical model and identified in simulations of a typical physical system. The problem of fluctuating Lyapunov exponents is expected to be common in simulations of higher-dimensional systems.

PACS numbers: 05.45.+b, 05.40.+j, 06.50.Dc

Physical theory is based on differential equation models. Computer simulation using the equations is often used to obtain information, such as long-term statistics, on the system being modeled. In climate modeling, for example, statistics of temperature and rainfall might be relevant. A basic requirement needed to interpret the result of a simulation is that the behavior of a solution determined by the computation, which is afflicted with small errors due to truncation and roundoff, is the same as the behavior of some true solution of the system under study. If there is a difference in behavior between computed solutions and actual solutions, the investigator cannot proceed. A climate model that continues to repeat winter conditions all year long because of accumulated numerical errors will not be useful for computing the mean yearly temperature.

This problem is especially acute when the system is chaotic. In that case, trajectories exhibit sensitive dependence on initial conditions: two trajectories with initial conditions that are extremely close tend to diverge exponentially from one another. Because of this effect, a small truncation or rounding error made at any step during the computation will tend to be greatly magnified by future evolution of the system. In view of this, it is natural to ask under what conditions the computed trajectory will be close to a true trajectory of the model.

Previous work on this topic [1–5] has resulted in computational techniques for "verifying" computer-generated trajectories for low-dimensional chaotic systems—that is, to produce a computer-assisted proof of the existence of a true trajectory of the system, called a shadowing trajectory, that closely tracks the computer-generated pseudotrajectory. (Even in this case, the statistical properties of the system may not be decided.)

Despite these positive results, not every pseudotrajectory can be shadowed. We believe that in systems with high-dimensional chaos, trajectories with intrinsic noise, such as computer-generated pseudotrajectories, can be shadowed only for short times. Consideration of simple examples of nonlinear maps [3] illustrates that there are critical points of trajectories where roundoff error or other noise, perhaps introduced at a distant part of the trajectory, can introduce new behavior. At such "glitches" all true trajectories diverge from the pseudotrajectory. In this case, when there is no true trajectory that stays near the pseudotrajectory, we say that the pseudotrajectory is unshadowable. There is no way known to ensure that a given computer simulation is representative of a true trajectory of the system (or even visits the entire phase space attractor) except when a proof of its shadowability is available. If the intrinsic noise is being injected by the environment itself rather than by a truncation error of a computer simulation, unshadowability raises interesting questions about the validity of deterministic modeling for the system.

In this Letter we describe a cause of unshadowable pseudotrajectories that is likely to occur widely in higher-dimensional chaotic dynamical systems. We will say that a Lyapunov exponent of a trajectory "fluctuates about zero" if for any positive number T the time-T Lyapunov exponent spends arbitrarily long stretches of the trajectory being positive and arbitrarily long stretches being negative. The finite-time Lyapunov exponents quantify the expansion and contraction of phase space along the trajectory over a time span of T. We will show that the existence of even one Lyapunov exponent which fluctuates about zero causes computer-generated trajectories to

0031-9007/94/73(14)/1927(4)$06.00

be unshadowable. Since a positive Lyapunov exponent is the signature of chaotic dynamics, this is a fact of critical importance to researchers studying the existence of chaos in computer models.

The manner in which a Lyapunov exponent fluctuating about zero leads to unshadowable pseudotrajectories is illustrated well by a theoretical model studied by Abraham and Smale [6] in 1970. In this example, there is an invariant set containing two fixed points: one with a single local expanding direction and one with a two-dimensional local expanding set. Typical trajectories wandering through the invariant set spend arbitrarily long times near each of the fixed points. The second largest Lyapunov exponent of such a trajectory is positive in trajectory segments near one and is negative in trajectory segments near the other, so this exponent fluctuates about zero. A ball of initial conditions beginning near the first fixed point will be squeezed into a line segment (with small thickness) under evolution of the dynamics. A computer-generated trajectory beginning in the ball, with truncation error δ, will be displaced a distance of δ from the line segment. When the region around the numerical trajectory develops a second expanding direction by visiting a neighborhood of the second fixed point, the numerical trajectory will be pushed away exponentially fast from the line segment of true trajectories, resulting in an unshadowable trajectory.

Although this example is nonphysical, we have found similar behavior in models of typical mechanical systems such as the double rotor [7]. For certain parameter settings, the double rotor has a chaotic attractor whose second largest Lyapunov exponent fluctuates about zero. As we discuss below, this effect causes almost every moderately long numerical trajectory to be unshadowable.

In order to quantify the phenomenon of unshadowability, we introduce the ideas of continuous shadowability and brittleness. A *continuously shadowable* pseudotrajectory is a computer-generated trajectory that can be continuously deformed into a true trajectory in such a way that the errors at each trajectory point are decreased monotonically to zero. Although this appears to be a stronger requirement than for a shadowable pseudotrajectory, it turns out that the hypotheses of the original Anosov-Bowen shadowing theorems [8–10], as well as computer-assisted shadowing methods mentioned above, imply not only that pseudotrajectories are shadowable but that they are continuously shadowable. We argue that continuous deformability to a true trajectory is a minimum requirement for accepting a computed simulation as meaningful information.

A fundamental phenomenon connecting continuously shadowable pseudotrajectories to their shadowing (true) trajectories is the existence of a constant of proportionality between the error magnitude of the pseudotrajectory and the distance the pseudotrajectory must move in phase space to be deformed into a true trajectory. We call this constant of proportionality the *brittleness* of the pseudotrajectory. An obvious necessary condition for continuous shadowability is that the brittleness times the error magnitude of the pseudotrajectory is smaller than the extent of the attractor in phase space. This leads to a practical algorithm for investigating the shadowability of a computer-generated trajectory. Using Jacobian information available from the simulation, it is possible to calculate a first-order approximation for the brittleness (the *test brittleness*) using small deformations of the pseudotrajectory. Knowledge of the test brittleness is a useful diagnostic for continuous shadowability of the pseudotrajectory or lack thereof.

Let f denote the map which represents one time step of the dynamics. For example, it may represent the time-T map produced by an ordinary differential equation (ODE) solver with one-step truncation error bounded by δ. (By one-step error, or noise, we mean the discrepancy between a time-T step of the ODE solver and a true time step of the differential equation, starting from the previous point.) If the present time is t_0 and the present state of the dynamical system is p_0, then the correct state at time $t_0 + T$ is $f(p_0)$. The ODE solver will produce p_1, where $|p_1 - f(p_0)| < \delta$. Then p_0 and p_1 are two points of a δ pseudotrajectory of the dynamical system f. Further integration of the simulation of f results in a δ pseudotrajectory $\{p_0, \ldots, p_N\}$ of length $N + 1$, where $|p_{i+1} - f(p_i)| < \delta$ for $i = 0, \ldots, N - 1$.

It would be desirable to know the existence of a true trajectory $x_0, \ldots, x_N$ [that is, $f(x_i) = x_{i+1}$ for $i = 1, \ldots, N - 1$] that ε shadows the pseudotrajectory. The true trajectory $\{x_i\}_{i=0}^{N}$ is said to be an *ε-shadowing trajectory* for the pseudotrajectory $\{p_i\}_{i=0}^{N}$ if $|x_i - p_i| < \varepsilon$ for $i = 0, \ldots, N$. Because of the exponential divergence of trajectories, if a shadowing trajectory exists, then the initial condition x_0 will differ from p_0. In fact, the first point x_0 of the true shadowing trajectory is expected to be found along the unstable direction emanating from p_0.

The shadowing lemma of Anosov [8] and Bowen [9] is a theoretical result for dynamical systems with hyperbolic structure. An attractor is called *hyperbolic* if the tangent spaces at each point of the attractor can be decomposed into uniformly expanding and contracting subspaces, such that the angle between these subspaces is bounded away from zero. The shadowing lemma states that for each nonzero distance ε, there exists an error magnitude δ such that each δ pseudotrajectory can be ε shadowed. Furthermore, under the hyperbolicity assumption, each such pseudotrajectory can be continuously shadowed within ε. (See, for example, the proof of the shadowing lemma in [10].)

We say that a pseudotrajectory has a *glitch* at point n if $\{p_i\}_{i=0}^{n}$ can be continuously shadowed, but $\{p_i\}_{i=0}^{n+1}$ cannot. One type of glitch is caused by a Lyapunov exponent fluctuating about zero, as described above. A schematic representation of a second type of glitch is shown in Fig. 1. Assume that q is a fixed point. Assume

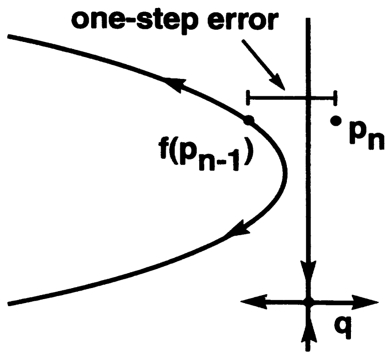

FIG. 1. A near tangency from a nonhyperbolic system. A small error in the computation of $f(p_{n-1})$ can push p_n across a stable manifold, resulting in a glitch.

that the pseudotrajectory has no error until iterate n, when error pushes p_n across the stable manifold of q. True trajectories must follow the unstable manifold of p_{n-1}, which separates exponentially from the stable manifold of q, so that no continuous shadowing trajectory can exist.

When a pseudotrajectory is continuously moved to a true trajectory by deforming its noise to zero, we will consider the distance moved by the pseudotrajectory to be the maximum distance any single trajectory point was moved. We call this the *shadowing distance.* The brittleness is the constant of proportionality between the shadowing distance of the pseudotrajectory and the original magnitude of the one-step error. The proportionality holds over a large range of noise levels, as long as the pseudotrajectory itself is not significantly changed. The brittleness is independent of the error magnitude (for small one-step errors) but depends on the error directions—a different set of one-step errors of the same magnitudes would, in general, lead to a different proportionality constant. The brittleness should be defined to be the maximum of this magnification factor found over all possible error directions.

The brittleness of a pseudotrajectory is a measure of its inability to be shadowed. If a pseudotrajectory is created with noise level 10^{-10} for a chaotic attractor of unit size, and if its brittleness is greater than 10^{10}, then one cannot expect a true trajectory closely shadowing the pseudotrajectory. For hyperbolic systems, pseudotrajectories of infinite length have finite (although possibly very large) brittleness [10]. For nonhyperbolic systems, one typically finds the brittleness to be an increasing function of the orbit length. As the length of a trajectory of a nonhyperbolic chaotic process increases, the brittleness grows as the trajectory gets increasingly close to nonhyperbolic regions of the dynamics. The expected length between glitches is therefore related to the amount to hyperbolicity possessed by the system.

Although the brittleness cannot be computed exactly without knowing the true trajectory, the test brittleness is a first-order approximation to this constant that is explicitly computable. The test brittleness is determined from the Jacobians (first derivative matrices) at the points p_i. Assuming the error at step i of the computation is δ_i, we want to approximate the correction c_i such that $f(p_i + c_i) = p_{i+1} + c_{i+1}$ for $i = 0, \dots, N-1$; write c_i as the sum $c_i = s_i + u_i$ of components in the stable (contracting) and unstable (expanding) directions at p_i; set $s_0 = u_N = 0$; and recursively solve

$$s_{i+1} = S(Df(p_i)s_i + \delta_i) \quad (1)$$

and

$$u_i = U(Df(p_{i+1})^{-1}u_{i+1} - \delta_i), \quad (2)$$

for s_i and u_i, $i = 0, \dots, N-1$, where S and U denote projection onto the stable and unstable directions, respectively. Then the ratio of the maximum magnitude of correction c_i to the magnitude of one-step error δ_i is the test brittleness. Since the computation is linear in δ_i, it is possible to choose the vectors δ_i to have magnitude 1. The δ_i are typically chosen to have randomly varying directions.

The double rotor map [7] is a four-dimensional map which descirbes the time evolution of a mechanical system consisting of two connected massless rods. The first rod rotates around a fixed pivot; the second rod pivots around the opposite end of the first rod. There is a mass at the free end of the first rod and equal masses at the ends of the second rod. A delta-function vertical impulse $f(t)$, of magnitude ρ, is applied to one of the ends at a constant time interval, at which the system's four phase variables (the two angular positions and momenta) are recorded. Interesting dynamical behavior is exhibited by this system for various settings of the parameter ρ

Application of the test brittleness algorithm to computer simulations of the double rotor are shown in Fig. 2. The difference between the shadowable case (parameter $\rho = 9$) and the unshadowable case ($\rho = 8$) is clear from this figure. In the vertical axis of Fig. 2(a), we graph the test brittleness of a length 10 000 pseudotrajectory for $\rho = 9$, created with one-step error magnitude 10^{-10}. The vertical extent of the graph is about 10^7, which is the test brittleness for this trajectory. We should then expect the shadowing distance to be about 10^{-3}. Figure 2(b) shows the same information for a typical pseudotrajectory of the double rotor with $\rho = 8$. The test brittleness in this case is seen to be greater than 10^{40}. This leads us to the prediction that at a minimum, 40 decimal digits of accuracy will be needed per iteration step in order to shadow a typical length 10 000 trajectory of the double rotor with $\rho = 8$. The brittleness of a pseudotrajectory increases with its length. For an orbit of length 10^5, our estimate of the test brittleness is 10^{100}.

The explanation of the difference in shadowability for the cases $\rho = 8$ and $\rho = 9$ lies in the different degrees of hyperbolicity of the two systems. A numerical study of the behavior of finite-time Lyapunov exponents

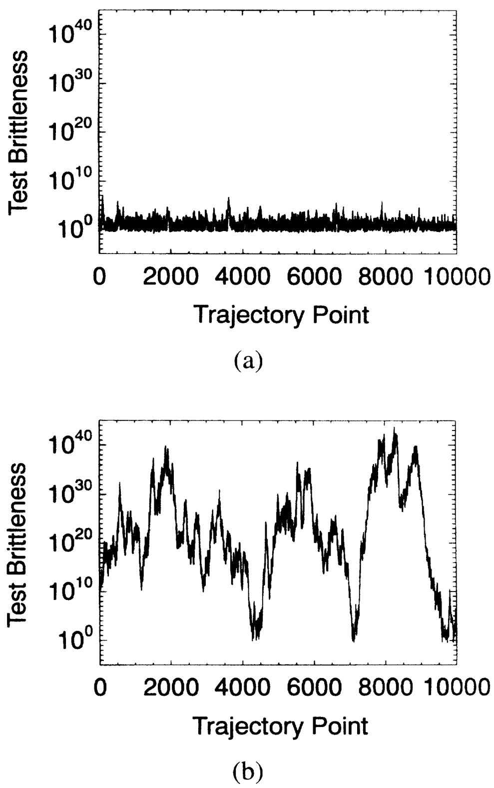

FIG. 2. First-order approximation of the shadowing distance per unit one-step error or brittleness as a function of trajectory point. The test brittleness is the vertical range of the graph. Results are shown for a 10 000 point trajectory of the double rotor with parameters (a) $\rho = 9$ and (b) $\rho = 8$.

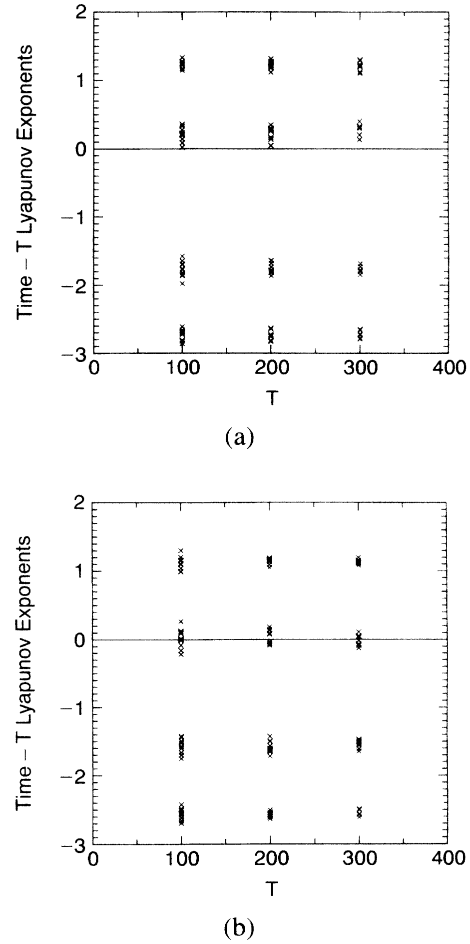

FIG. 3. Estimates of the four time-T Lyapunov exponents of the double rotor, for $T = 100$, 200, and 300. A dozen simulations were done for each T. (a) In the $\rho = 9$ case, there is consistently only one positive finite-time Lyapunov exponent. (b) For $\rho = 8$, trajectory segments alternate between one-dimensional and two-dimensional expanding subspaces.

for the two parameter values of the double rotor is shown in Fig. 3. For $\rho = 8$, the finite-time Lyapunov exponents show fluctuation between one and two positive exponents. The second largest exponent fluctuates about zero. For $\rho = 9$, there are consistently two positive exponents.

A close examination of the dynamics of the double rotor reveals an explanation for the fluctuating number of positive finite-time Lyapunov exponents in the $\rho = 8$ case. There are many periodic orbits embedded in the attractor whose local behavior varies in a qualitative way. Some of the periodic points have one expanding direction and three contracting directions, while others have two expanding and two contracting [7]. As a trajectory (or pseudotrajectory) moves densely through the attractor, its number of positive finite-time Lyapunov exponents varies as it moves among the varying type of periodic orbits. Although the double rotor with $\rho = 8$ is the first physical example for which this behavior has been demonstrated, we expect it to be commonly found in higher-dimensional systems.

This research was supported in part by grants from the NSF (Computational Mathematics and Physics programs) and DOE (Office of Scientific Computing).

[1] C. Grebogi, S. Hammel, and J. Yorke, J. Complexity **3**, 136 (1987); Bull. Am. Math. Soc. **19**, 465 (1988).
[2] C. Grebogi, S. Hammel, J. Yorke, and T. Sauer, Phys. Rev. Lett. **65**, 1527 (1990).
[3] T. Sauer and J. Yorke, Nonlinearity **4**, 961 (1991).
[4] S.-N. Chow and K. Palmer, Dynamics and Diff. Eq. **3**, 361 (1991); J. Complexity **8**, 398 (1992).
[5] J.M. Sanz-Serna and S. Larsson, Appl. Num. Math. **13**, 181 (1993).
[6] R. Abraham and S. Smale, Proc. Symp. Pure Math. **14**, 5 (1970).
[7] C. Grebogi, E. Kostelich, E. Ott, and J.A. Yorke, Physica (Amsterdam) **25D**, 347 (1987); R. Romeiras, C. Grebogi, E. Ott, and W.P. Dayawansa, Physica (Amsterdam) **58D**, 165 (1992).
[8] D.V. Anosov, Proc. Steklov Inst. Math. **90**, 1 (1967).
[9] R. Bowen, J. Diff. Eq. **18**, 333 (1975).
[10] K. Meyer and G.R. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-body Problem* (Springer-Verlag, New York, 1992).