Flux Creep in Superconducting Films: An Exact Solution

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An exact solution of the nonlinear integral equation is obtained that describes flux creep in strips or disks. We calculate both analytically and numerically the relaxation of the electric field $\mathbf{E}(\mathbf{r},t)$, sheet current $\mathbf{J}(\mathbf{r},t)$, flux density $\mathbf{B}(\mathbf{r},t)$, and magnetic moment $M(t)$. After some transient period, $\mathbf{E}(\mathbf{r},t)$ approaches a universal nonmonotonic shape, regardless of the detailed form of the current-voltage law $E(J)$. We predict that, in spite of the relaxation of M, $\mathbf{B}(\mathbf{r},t)$ will both increase and decrease with t in regions separated by "neutral lines," along which $\mathbf{B}(\mathbf{r},t)$ *does not depend on time.*

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The critical state of superconducting films in perpendicular magnetic field $\mathbf{H}$ exhibits a number of novel features [1–8] as compared to the Bean model for slabs in a parallel field [9]. Recently this problem has attracted much interest since new *static* analytical solutions for the current $\mathbf{J}(\mathbf{r})$ and magnetic field $\mathbf{H}(\mathbf{r})$ in perpendicular field were obtained [5–8] which are qualitatively different from the well known parallel case. At the same time, high resolution scanning Hall probe [10–12] and magneto-optic [5,13,14] measurements revealed nontrivial features of $\mathbf{H}(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ in thin high-T_c superconductors which are important for the correct interpretation of magnetization curves and flux pinning in the perpendicular field orientation usually used in experiments.

As for the flux *dynamics* in thin films in perpendicular field, a detailed theoretical description has been given so far only for the linear Ohmic regime [15], which occurs in high fields H at temperatures T above the irreversibility line. This analysis has shown that even the linear flux dynamics in thin films differs qualitatively from the parallel case, since the equation for $\mathbf{H}$ turns out to be nonlocal [15]. This effect is essential for the calculation of the linear ac response of thin films, for example, the complex magnetic susceptibility or the attenuation and frequency change of vibrating superconductors [16,17]. Below the irreversibility line, the situation is complicated by the strong nonlinearity of the E-j characteristics below the critical current density j_c, where the electric field $E(j)$ caused by thermally activated drift of vortices can be written in the form

$$E = E_c \exp[-U(j)/T]\,. \tag{1}$$

Here $U(j)$ is a flux-creep potential barrier which vanishes at $j = j_c$, and E_c is a crossover electric field which defines j_c by $E(j_c) = E_c$. Formula (1) together with the Maxwell equations completely determines the dynamics of the flux creep, which can be formulated in terms of a nonlinear diffusion of the magnetic flux through the sample [18]. For the perpendicular field orientation, this nonlinear flux diffusion also becomes *nonlocal* and is described by a nonlinear integral equation obtained in Ref. [15]. In this Letter we obtain exact solutions of this nonlinear and nonlocal equation which describe flux creep in a thin strip and a circular disk.

We first consider a superconducting strip placed in a uniform time-dependent magnetic field $H_a(t)$ parallel to the z axis, the strip being infinite along the x axis and having a width $2a$ along the y axis and thickness $d \ll a$. The electric field $\mathbf{E}(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$ have then only x components $E(y,t)$ and $j(y,t)$. The z component of $\mathbf{H}(y,t)$ at $z=0$ is given by Ampère's law, which to an accuracy of $d/a \ll 1$ reads

$$H(y) = H_a(t) + \frac{1}{2\pi}\int_{-a}^{a} \frac{J(u)\,du}{y-u}\,. \tag{2}$$

Here $J(y) = \int j(y,z)\,dz$ is the sheet current; we shall assume here $J = jd$. To obtain a self-consistent equation for $E(y)$, we substitute Eq. (2) into the Maxwell equation $\mu_0 \partial H/\partial t = -\partial E/\partial y$, writing $\partial J/\partial t = (\partial J/\partial E)\partial E/\partial t$. This gives

$$\frac{1}{\mu_0}\frac{\partial E}{\partial y} = -\frac{\partial H_a}{\partial t} - \frac{1}{2\pi}\int_{-a}^{a} \frac{\partial J}{\partial E}\frac{\partial E(u,t)}{\partial t}\frac{du}{y-u}\,. \tag{3}$$

The integro-differential equation (3), which describes the nonlinear flux diffusion in the strip in perpendicular field, is *nonlocal*, unlike the local equation $E'' = \mu_0(\partial j/\partial E)\dot{E}$ for the parallel case (the prime and overdot denote space and time derivatives, respectively). Notice that these local or nonlocal diffusion equations can be written in terms of different variables $\mathbf{H}(\mathbf{r},t)$, $\mathbf{J}(\mathbf{r},t)$, or $\mathbf{E}(\mathbf{r},t)$. We use here the latter representation since it turns out that the time evolution of $\mathbf{E}(\mathbf{r},t)$ is universal for different models of thermally activated flux creep [19], while the more model-dependent $\mathbf{J}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are obtained by inserting the universal $\mathbf{E}(\mathbf{r},t)$ into the specific E-j characteristics. As an illustration, we consider a vortex glass and/or collective creep model for which $U(j) = [(j_c/j)^\beta - 1]U_0$, where U_0 is a characteristic acti-

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vation energy and $\beta > 0$ [20,21]. In this case the differential conductivity $\partial j/\partial E$ in Eq. (3) is given by

$$\frac{\partial j}{\partial E} = \frac{T j_c}{U_0 \beta E}\left[1 + \frac{T}{U_0}\ln\frac{E_c}{E}\right]^{-1-1/\beta}. \quad (4)$$

Since the ratio T/U_0 is much smaller than unity well below the irreversibility line, we have $\partial j/\partial E = j_1/E$ over a wide region of E, except for exponentially small fields $E < E_g \approx E_c \exp[-U_0\beta/(1+\beta)T] \ll E_c$ (here $j_1 = Tj_c/\beta U_0$ is the apparent flux creep rate $dj/d\ln t$). But even at $U_0\beta/(1+\beta)T \approx 1$, the logarithmic terms in Eq. (4) give rise to only slowly varying corrections to the universal dependence $dj/dE \sim 1/E$, which virtually results from the thermally activated character of the flux dynamics at $j < j_c$ and takes place for any power dependence of $U(j)$ in Eq. (1) [19].

For this reason, we consider the case $\partial j/\partial E = j_1/E$ in more detail, writing Eq. (3) in the form

$$E'(y,t) = -\mu_0 \dot{H}_a(t) - \frac{\mu_0 d j_1}{\pi}\int_0^a \frac{\dot{E}(u,t)}{E(u,t)}\frac{u\,du}{y^2-u^2}. \quad (5)$$

Here we have used the odd symmetry of the magnetization currents $J(y) = -J(-y)$ in the absence of a transport current. From Eq. (5) it is immediately seen that a linearly increasing magnetic field $H_a(t) = \dot{H}_a t$ induces a steady-state electric field $E = -\mu_0 \dot{H}_a y$ across the strip. When the increase of $H_a(t)$ is stopped, the field $E(y,t)$ begins to decay with t due to the nonzero resistivity of a superconductor at $j < j_c$ caused by thermal activation.

Remarkably, the relaxation of $E(y,t)$ at $\dot{H}_a = 0$ can be obtained *exactly* from Eq. (5) by the ansatz $E(y,t) \sim f(y)g(t)$, with the functions $f(y)$ and $g(t)$ depending only on y and t, respectively. With this ansatz, $f(u)$ cancels in the integrand in Eq. (5), and $f(y)$ is obtained by a simple integration of (5), provided $g(t)$ obeys the equation $\dot{g} = -g^2$. Therefore, $g(t) = 1/(t+\tau)$ with τ an integration constant, and the exact solution of (5) is

$$E(y,t) = \frac{\mu_0 j_1 a\, d}{2\pi(t+\tau)} f_{\text{strip}}\left(\frac{y}{a}\right), \quad (6)$$

$$f_{\text{strip}}(\eta) = (1+\eta)\ln(1+\eta) - (1-\eta)\ln(1-\eta) - 2\eta\ln|\eta|. \quad (7)$$

In terms of the integral kernel $K(y,u) = \ln|(y-u)/(y+u)|$ of Ref. [15], one has $f_{\text{strip}}(\eta) = -\int_0^1 K(\eta,u)\,du$. Our method is easily generalized to varying thickness $d(y)$.

For a circular disk with radius a, one obtains the same exact solution (6) if $f_{\text{strip}}(y/a)$ is replaced by a similar universal function $f_{\text{disk}}(r/a)$ obtained by integrating the disk kernel $Q(r,u)$ [15], $f_{\text{disk}}(\eta) = -\int_0^1 Q(\eta,u)\,du$, or

$$f_{\text{disk}}(\eta) = \int_0^1 du\,\frac{u}{\eta}\int_0^{\eta/u}\left[\frac{E(k)}{1-v} + \frac{K(k)}{1+v}\right] v\,dv, \quad (8)$$

where $E(k)$ and $K(k)$ are complete elliptical integrals and $k^2 = 4v/(1+v)^2$. The function (8) is fitted with high precision (absolute deviation $< 2.9\times10^{-4}$) by

$$f_{\text{disk}}(\eta) \approx c_1\eta + c_2\eta^2 + c_3\eta\ln\eta + c_4(1-\eta)\ln(1-\eta), \quad (9)$$

with $c_1 = 0.45880$, $c_2 = 0.37313$, $c_3 = -1.55714$, and $c_4 = -0.97479$ (Fig. 1).

Formulas (6) and (7) are exact for the linearized flux-creep barrier $U(j)/T = (j - j_c)/j_1$ for which $\partial j/\partial E = j_1/E$ and $J(y,t) = J_c - J_1\ln[E_c/E(y,t)]$. These results, however, are more general and hold for a wide class of nonlinear $U(j)$ as well. Figure 1 shows the results of numerical time integration of Eq. (3) (described in detail in Ref. [15]) for power laws $E(j) \sim j^n$ with various n. As seen from Fig. 1, the relaxation leads to distributions $E(y,t)$ which are indeed very close to the universal profiles (6) and (7) at $n \gg 1$ with $E \sim 1/(t+\tau)^{n/(n-1)}$ [19]. Even the comparatively "weak" nonlinearity $E \sim j^3$ yields this universal critical state. Only for the Ohmic case $E = \rho j$, or the Kim-Anderson model [22] $E \sim \sinh(j/j_1)$ which becomes linear at $j \ll j_1$, the profiles of E and J attain a different, also universal form (Fig. 1) and decay exponentially, $E(y,t) = \rho j(y,t) \sim f_0(y/a)\exp(-t/\tau_0)$. Here $f_0(y/a)$ and τ_0 are the lowest eigenfunction and eigenvalue of the linearized Eq. (3).

We consider the effect of nonlinearity of $U(j)$ on the

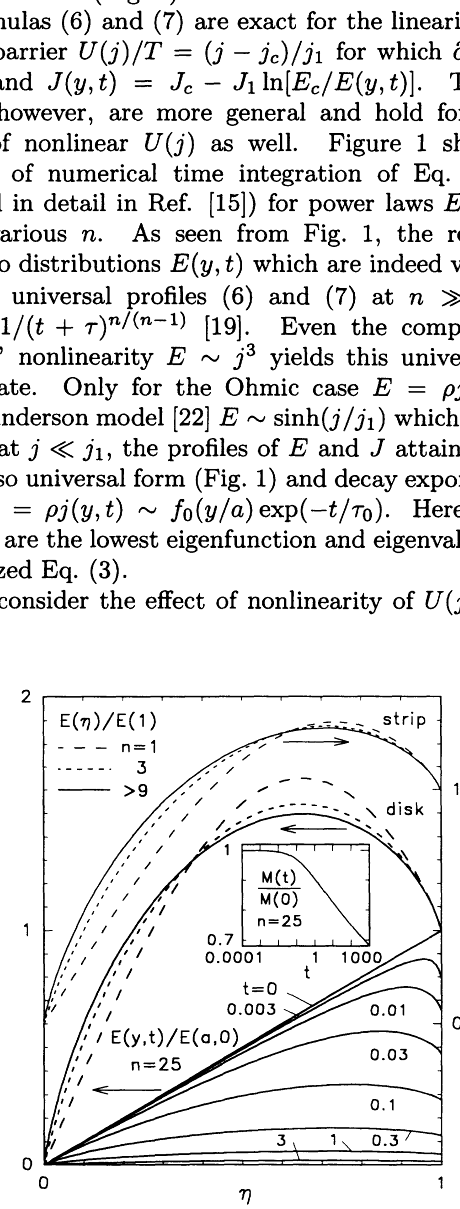

FIG. 1. Top: Universal profiles which the electric field $E(\mathbf{r},t)$ approaches during flux creep at $t \gg \tau$. Shown is $E(\eta,t)/E(1,t)$ for strips of half width a and for disks of radius a versus $\eta = y/a = r/a$ obtained numerically for the model $E(j) = E_c(j/j_c)^n$ for $n = 1, 3$, and ∞. The curve $n = \infty$ (bold line) is reached at $n \geq 9$ and coincides with $f_{\text{strip}}(\eta)/f_{\text{strip}}(1)$ (7) or $f_{\text{disk}}(\eta)/f_{\text{disk}}(1)$ (9). Bottom: Relaxation of $E(y,t)$ for a strip with $n = 25$ at times $t = 0, 0.003, 0.01, 0.03, 0.1, 0.3, 1$, and 3 in units $adj_c\mu_0/2\pi E_c \approx nt_0$ (16) ($j_1 = j_c/n$). The inset shows the relaxing magnetization $M(t)$ for this case, which approaches $M(t) - M(0) \sim 1/t^{1/24} - 1 \sim -\ln t$.

universal behavior of $E(y,t)$ for the power law $U(j)$ with $\partial j/\partial E$ given by Eq. (4). As follows from (3)–(5), this gives rise to an additional factor

$$F(u,t) = \{1 + (T/U_0)\ln[E_c/E(u,t)]\}^{-1-1/\beta}$$

in the integrand of Eq. (5). Furthermore we note that at $T \ll U_0$ the contribution from the coordinate part $f(y)$ of $E(u,t)$ in the slowly changing logarithmic term in $F(u,t)$ may be replaced by its mean value, after which F becomes independent of u and may be accounted for by using an effective time $\tilde{t}$ instead of t in (6),

$$\tilde{t} = \int_0^t [1 + (T/U_0)\ln(t'/t_0)]^{1+1/\beta} dt', \quad t \gg t_0, \qquad (10)$$

where $t_0 = c\mu_0 j_1 ad/E_c$, and $c \approx 1$ is a constant which results from the averaging of $\ln f(y)$ in $F(u,t)$. As seen from Eq. (10), the difference between t and $\tilde{t}$ manifests itself only at exponentially large times $t > t_c \approx t_0 \exp[U_0\beta/(1+\beta)T]$.

Substituting Eqs. (6), (7), and (10) into the Maxwell equation $\mu_0 \dot{H} = -E'$ and integrating over t, we obtain $H(y,t)$ at $t \gg t_0$,

$$H(y,t) = H_a(0) - \frac{J_c \ln(a^2/y^2 - 1)}{2\pi[1 + (T/U_0)\ln(t/t_0)]^{1/\beta}}. \qquad (11)$$

For $(T/\beta U_0)\ln(t/t_0) \ll 1$, formula (11) reduces to the result we derived for the exponential $E(j)$. As in the linear case [15], the singularity in $H(y)$ at the strip edges $|y| = a$ is suppressed by the finite film thickness, and the logarithm in (11) attains a maximum $\approx \ln(a/2d)$. The singularity at $y = 0$ is also fictitious since the averaging of $\ln f(u)$ in $F(u)$ becomes invalid at $y = 0$ because $F(u)$ vanishes at the same value $u = 0$ at which the kernel of Eq. (3) is singular. If we account for the vanishing of $E(u)$ in $F(u)$ at $u = 0$, the weak logarithmic divergence of $H(y \to 0)$ disappears. The relaxation of $H(y,t)$ described by Eq. (11) can be seen in Fig. 2.

We now consider the relaxation of the magnetic moment of the strip, $M(t) = 2\int_0^a yJ(y,t)\,dy$ (per unit length; the factor 2 accounts for the U turn of the magnetization current at the ends of the strip [7]), or of the disk, $M(t) = \pi\int_0^a r^2 J(r,t)\,dr$. Notice that although Eq. (6) is an exact solution of (5), it does not satisfy the initial condition $E(y,0) = -\mu_0 \dot{H}_a y$. The transient period during which $E(y,t)$ depends on the previous history of $H_a(t<0)$ is described by the time constant τ in (6) [19]. We shall use Eq. (6) for a self-consistent calculation of $M(t)$ by choosing a τ value which ensures the correct value of M at $t = 0$ [19]. From (6) we get for the strip

$$M(t) = a^2\left[J_c + J_1 \ln\frac{\mu_0 j_1 ad}{2\pi\tau E_c} + \alpha J_1\right] - M_1\ln(1 + t/\tau), \qquad (12)$$

with $M_1 = a^2 J_1$ and $\alpha = 2\int_0^1 \eta \ln f_{\rm strip}(\eta)\,d\eta = 0.450$.

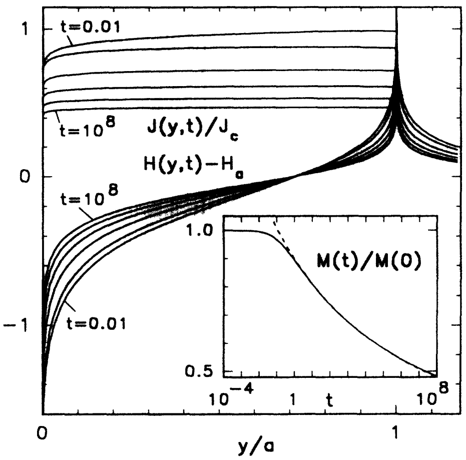

FIG. 2. Current density $J(y,t)/J_c$ and magnetic field $H(y,t) - H_a$ in a strip for the vortex glass and/or collective creep model (1), (4) with $T/U_0 = 1/19$ and $\beta = 1$ at times $t = 0.01, 1, 100, 10^4, 10^6$, and 10^8. Units are $J_c = j_c d$ for field and sheet current J, and $ad\mu_0 j_c/2\pi E_c \approx 12t_0$ (16) for time. Flux creep starts at $t = 0$ with $E(y,t) = E_c y/a = -\mu_0 \dot{H}_a$. The inset shows the relaxing magnetization for this model, which for $t > 1$ is fitted extremely well by (17) (dashed line).

For the disk, one should replace in (12) the prefactor a^2 by $\pi a^3/3$ and one has $M_1 = (\pi a^3/3)J_1$ and $\alpha = 3\int_0^1 \eta^2 \ln f_{\rm disk}(\eta)\,d\eta = 0.110$. The value $M(0)$ is calculated by taking $E(y,0) = -\mu_0 \dot{H}_a y$, whence [19]

$$M(0) = a^2\left[J_c + J_1 \ln\frac{\mu_0 \dot{H}_a a}{E_c} - \frac{J_1}{2}\right]. \qquad (13)$$

Equating the expressions in the square brackets in (12) and (13) we finally get

$$M(t) = M(0) - M_1 \ln\left(1 + \frac{t}{\tau}\right), \quad \tau = 0.41\,\frac{j_1 d}{\dot{H}_a}. \qquad (14)$$

Formulas (13) and (14) are similar to those for a slab in parallel field [19]: In both cases τ is proportional to the minimum sample size and inversely proportional to the ramp rate $\dot{H}_a$. In the regime of the steady-state relaxation $t \gg \tau$ the value τ cancels in (12); thereby $M(t)$ becomes independent of the initial conditions,

$$M(t) = M_c - M_1 \ln(t/t_0), \qquad (15)$$

$$M_c = a^2 J_c, \quad t_0 = a\,d\,\mu_0 j_1/4E_c, \qquad (16)$$

where both J_c and t_0 depend on the voltage criterion E_c.

The decay of $M(t)$ in the vortex glass/collective creep model is described by the interpolation formula [21]

$$M(t) = M_c/[1 + (T/U_0)\ln(t/t_0)]^{1/\beta}, \quad t \gg t_0. \qquad (17)$$

This time dependence, which coincides with that of $H - H_a$ (11) and of J, describes the numerical results for strips (Fig. 2) and disks very accurately (deviation

$< 0.5\%$) for $5 < t/t_0 < 10^{40}$. The fit of $M(t)$ is improved (deviation $< 0.2\%$) if in (17) t_0 (16) is replaced by $1.2t_0$ for the strip and by $0.8t_0$ for the disk. Interestingly, the more correct replacement in (17) of t by $\tilde{t}$ (10) does not improve this fit noticeably.

Despite an apparent similarity of Eqs. (13)–(17) for the parallel and perpendicular cases, these correspond to qualitatively different regimes of local and nonlocal flux diffusion, respectively. For instance, the nonlocality manifests itself in the fact that the value t_0 in perpendicular field turns out to be proportional to the cross-sectional area ad of strips or disks, whereas in parallel field one has $t_0 \sim a^2$ [19] for slabs of thickness $2a$. The same geometric dependencies are valid for the linear relaxation times τ_0 of Ohmic conductors [15]. More striking differences become visible when comparing the electric field profiles (6) with the parabolic distribution $E(y,t) = \mu_0 j_1[2ay - y^2\mathrm{sgn}(y)]/2(t+\tau)$ for the parallel case [19]. For instance, the electric field $E(y,t)$ has logarithmically diverging slope $E'(y,t)$ at the edges of the strip, the profile being *nonmonotonic* with the maximum at $y = y_m = a/\sqrt{2}$ (Fig. 1). This maximum gives rise to anomalous flux-creep dynamics determined by Eq. (11) (Fig. 2). Namely, in the central region $y < y_m$, the local field $H(y,t)$ increases with t due to the flux penetration into the strip. However, in the lateral regions $y_m < |y| < a$, the field *decreases* with t, in stark contrast to the parallel case. This decrease is due to the field enhancement caused at the edges by the large demagnetization factor. During the relaxation, flux creep smooths out the field profile $H(y,t)$ such that $\dot{H} > 0$ in regions where $H(y,t) < H_a$, and $\dot{H} < 0$ in regions where $H(y,t) > H_a$. As a result, there appear "neutral lines" $y = \pm a/\sqrt{2}$ in the strip, and a neutral circle $r = 0.652a$ in the disk, along which $H(\mathbf{r},t)$ remains constant despite the flux creep. This effect should be observable by magneto-optic or Hall probe experiments.

In summary, we considered flux creep in thin superconductors in perpendicular field. An exact solution of the nonlinear integral equation is obtained, which describes nonlocal diffusion in superconducting strips and disks. The presented method accounts for both the sample geometry and the strong nonlinearity of $E(j)$ in the flux-creep regime. For a wide class of nonlinear resistivities, the electric field profiles after some transient period take the universal nonmonotonic shapes (6)–(9) and decrease approximately as $1/t$. Anomalous flux dynamics in flat superconductors is predicted, with the coexistence of regions of decreasing and increasing flux density.

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