Large-N Chiral Field in Two Dimensions

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We present the exact and explicit solution of the principle chiral field model in two dimensions for infinitely large rank group. In particular we show that in the large-N limit the spectrum of the theory does not contain pointlike particles. The energy of the ground state as a fucntion of external "Noether" field and the beta function are explicitly found. The nonperturbative threshold behavior near the mass gap m is $f(h) \sim (h - m)/\ln(h - m)$, exhibiting a similarity with the 1D bosonic string theory.

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Recent progress in the understanding of lower dimensional theories is partially due to the advantage of discrete methods. The combinatorial methods of matrix models, as well as the continuous approaches, have appeared so far to be ineffective for higher dimensional ($D > 1$) target space. On the other hand, it is known for a long time that certain matrix field theories are completely integrable in two dimensions for an arbitrary size of the matrix field. One of the most representative integrable matrix field theories is the *principle chiral field*

$$S = \frac{N}{2\lambda_0} \int d^2x \, \mathrm{tr}[\partial_\mu g^\dagger \partial_\mu g], \qquad (1)$$

where g is an element of the, say, SU(N) group. Here we present the large-(N) solution of the model on the basis of the finite-(N) solution [1–4].

It turns out that the spectrum of the SU(N) model contains $N - 1$ massive particles. They transform under the diagonal of SU(N) ⊗ SU(N) and form multiplets of all fundamental representations of SU(N) algebra, the vector representation, and all the antisymmetric tensors according to the Dynkin diagram. The spectrum of masses is

$$m_l = m \frac{\sin[(\pi/N)\, l]}{\sin(\pi/N)}, \qquad (2)$$

where $l = 1, \ldots, N - 1$ is the rank of a fundamental representation and $m = m_l$ is the mass of the vector particle. In the two-loop approximation it is given by $m = \Lambda \lambda_0^{-1/2} \exp(-4\pi/\lambda_0)$ where Λ is a cutoff. All particles are bound states of the vector particles.

At large N we must distinguish two physically different situations: $N \to \infty$ but $m = m_l =$ fixed. This means that $m_l = l m_1$, so that the lth particle is not a bound state any more. This suggests that the interaction vanishes in this limit. Below we consider a more interesting limit: $N \to \infty$ but the heaviest mass $m_{N/2}$ of the largest antisymmetric tensor remains fixed. In this case the masses fuse so that the mass spectrum becomes continuous. The label running along the Dynkin diagram becomes a continuous parameter. We observe that an extra dimension emerges from the matrix structure of the field. This means that at $N \to \infty$ particles cannot be separated either in momentum or in coordinate space, so the theory ceases to be a theory of pointlike particles.

Quantum states of the model are characterized by the values of the conserved left current $L_l = \int dx \, \mathrm{tr} H_l g^{-1} \partial_0 g$ and right current $R_l = \int dx \, \mathrm{tr} H_l \partial_0 g g^{-1}$ where $H_l = \mathrm{diag}(0, \ldots 1, -1, \ldots 0)$ is a basis in the Cartan subalgebra. We shall find the energy of the ground state $\mathcal{E}(h)$ as a function of "Noether" field by adding, say, a term $Q_L = i h_l^L L_l$ or $Q_R = i h_l^R R_l$ or both of them to the Hamiltonian of the model (1). In what follows we will choose for simplicity $\vec{h}^R = \vec{h}^L = \vec{h}/2$. The parameters h_l play the role of a chemical potential for the elementary particles of the model, so the energy $\mathcal{E}(h)$ is the energy of the ground state with a symmetry of the Young tableau $[1^{\mathcal{N} - \mathcal{N}_\infty}, 2^{\mathcal{N}_1 - \mathcal{N}_2}, \ldots, N^{\mathcal{N}_{N-1}}]$ where $\mathcal{N}_{l-1} - \mathcal{N}_l = -d\mathcal{E}(h)/dh_l$ (at finite N this quantity in chiral models has been studied in Ref. [2,5]).

The large-N solution is explicit. It is given by Eqs. (15) and (16).

S-matrix and Bethe-ansatz equations for any N.—The most economical way to obtain Bethe-ansatz equations for the chiral field is the factorized bootstrap method [3,4], rather than direct diagonalization of the Hamiltonian of the model [1,2]. Below we give a sketch of this approach.

(1) S-matrix: The chiral field is renormalizable and asymptotically free. It is invariant under the left-hand and right-hand group transformations $G \otimes G$ and the action is expressed by elements of Lie algebra. Therefore it is natural to assume that the elementary particles are massive and belong to some fundamental representations of the diagonal of $G \otimes G$.

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Furthermore, the model is integrable; therefore the scattering is factorized. Under these assumptions the minimal S matrix (factorized scattering matrix with a minimal set of singularities) can be determined unambiguously. It turns out that once we assume that there is a particle in some, say, lth fundamental representation, the factorized bootstrap will tell us that there are particles in all $N-1$ fundamental representations. Therefore we may start from the vector particle. The factorized $SU(N) \otimes SU(N)$ scattering matrix for vector particles is the tensor product of the $SU(N)$ factorized vector S matrices $S = X(\theta)S(\theta) \otimes S(\theta)$. Here θ is a rapidity of relativistic particle ($p^0 = m\cosh\theta$, $p^1 = m\sinh\theta$) and $X(\theta)$ is the Castillejo-Dalitz-Dyson (CDD) factor which cannot be determined by factorization, unitarity, and crossing symmetry conditions. The $SU(N)$ factorized S matrix of vector particles is well known [6]. It is

$$S(\theta) = u(\theta)\left(P^+ + \frac{\theta + i2\pi/N}{\theta - i2\pi/N}P^-\right)$$

where $P^\pm$ is the projection operator onto symmetric (antisymmetric) states.

The amplitude in the symmetric channel $u(\theta)$ and the amplitude in the cross channel (particle-antiparticle scattering)

$$t(\theta) = \frac{\frac{1}{2} - \theta/2i\pi}{\frac{1}{2} - 1/N - \theta/2i\pi}u(i\pi - \theta)$$

obey unitarity conditions $t(\theta)t(-\theta) = u(\theta)u(-\theta) = 1$. The minimal solution of these equations is

$$u(\theta) = \frac{\Gamma(1 - \theta/2\pi i)\,\Gamma(1/N + \theta/2\pi i)}{\Gamma(1 + \theta/2\pi i)\,\Gamma(1/N - \theta/2\pi i)}. \tag{3}$$

Finally the CDD factor is chosen to cancel all double zeros and double poles on the physical sheet $0 < \mathrm{Im}\theta < \pi$:

$$X(\theta) = \frac{\sinh\left(\frac{1}{2}\theta + i\pi/N\right)}{\sinh\left(\frac{1}{2}\theta - i\pi/N\right)}.$$

This gives the S matrix of the vector particles. It has a pole on the physical sheet at $\theta_b = 2\pi i/N$ in the antisymmetric channel. The pole corresponds to the first bound state (the second rank antisymmetric tensor) with mass m_2. The S matrix of these particles can also be found by tensoring the vector S matrix (the fusion procedure). It also has a pole in the antisymmetric channel, and so on. In this way the whole mass spectrum (2) can be generated.

(2) Bethe-ansatz equations: The thermodynamic properties of the model can be obtained by imposing (periodic) boundary conditions. For an integrable problem they imply the balance of two particle scattering phases and a phase of a free motion between collisions. Consider for example a state with $\mathcal{N}$ vector particles in the box L where all particles have the same spin. Then for the ith particle with the momentum $m\sinh\theta_i$, the periodic boundary conditions lead to

$$\exp(imL\sinh\theta_\alpha) = \prod_{\beta=1,\neq\alpha}^{\mathcal{N}} \exp[i\phi(\theta_\alpha - \theta_\beta)], \tag{4}$$

where $\exp[i\phi(\theta)] = u^2(\theta)X(\theta)$ is the amplitude of the symmetrical channel. To obtain the Bethe-ansatz equation for the state with a more general Young tableau one has to consider complex rapidities of the bound states—"strings" $\theta^{r,(l)} \to \theta^{(l)} + 2r\pi i/N$, where θ^l is a rapidity of the lth particle and r is an integer running between $-l/2$ and $l/2$. Substituting this into Eq. (4) and multiplying equations over r we shall obtain the equations for the rapidities of the state which contains $\mathcal{N}_l$ particles of the kind l. Taking the logarithm of both sides of Eq. (4) we obtain

$$Lm_l\sinh\theta_\alpha^{(l)} = 2\pi J_\alpha^{(l)} + \sum_{n=1}^{N-1}\sum_{\alpha=1,\neq\beta}^{\mathcal{N}_n} \phi_{ln}\left(\theta_\alpha^l - \theta_\beta^{(n)}\right), \tag{5}$$

where $\phi_{ln}(\theta) = \sum_{|r|<l/2,|r'|<n/2}\phi(\theta + 2ri\pi/N + 2r'i\pi/N)$ is the scattering phase of the lth and the nth particles, and integers J are the quantum numbers of the states. The energy of this state is obviously

$$E = \frac{1}{L}\sum_{l=1}^{N-1} m_l \sum_{\alpha=1}^{\mathcal{N}_l}\cosh\theta_\alpha^{(l)}. \tag{6}$$

(3) Spectral equations: The next step is to find rapidities to minimize the energy (6) in the thermodynamic limit $\mathcal{N}_l/L = n_l$, while $L \to \infty$. We assume that in the ground state θ's are distributed smoothly between $-B_l$ and B_l with a distribution function $\rho_l(\theta)$. Then Eq. (5) implies the spectral equations

$$\frac{1}{2\pi}m_l\cosh\theta = \sum_n \int_{-B_l}^{B_l} R_{ln}(\theta - \theta')\rho_n(\theta')d\theta', \tag{7}$$

where $R_{ln}(\theta) = \delta_{ln} - (1/2\pi)d\phi_{ln}(\theta)/d\theta$. The Fermi rapidities B_l are determined by the number of particles in the lth representation: $\int_{-B_l}^{B_l}\rho_l(\theta)d\theta = n_l$. The energy of the state is then

$$E = \sum_l \int_{-B_l}^{B_l} m_l\cosh\theta\,\rho_l(\theta)d\theta. \tag{8}$$

An explicit form of the scattering kernel R_{ln} was found in [2]. Its Fourier transformation is

$$R_{ln}(\omega) = 2\frac{\sinh[\pi\omega(1 - 1/N)]\sinh(\pi\omega n/N)}{\sinh\pi\omega}$$

at $l > n$ and $R_{ln} = R_{nl}$.

Large-N solution.—At large N we can consider a particular distribution of fields h_l which creates all different particles on equal footing, namely, one which follows the spectrum of masses (2): $h_l = (h/m)m_l$. This field creates $n_l = n(m_l/m)$ particles in the lth representation (the most representative Young tableau). In this case all Fermi momenta are equal: $B_l = B$ and $\rho_l = (1/N)(m_l/m)\rho$. Then the spectral equations (7) can be easily diagonalized. They reflect the structure of the Cartan matrix and moreover have the same eigenvectors

$$\sum_{l,n=1}^{N-1} \chi_l^{(p)} R_{ln}(\omega) \chi_n^{(p')} = R^{(p)}(\omega)\delta^{p,p'},$$
$$R^{(p)}(\omega) = \frac{2N}{\pi} \sum_{r=-\infty}^{\infty} \frac{|\omega|}{\omega^2 + (p + rN)^2}, \tag{9}$$

where $\chi_l^{(p)} = \sqrt{2/N}\sin(\pi pl/N)$, $p = 1, 2, \ldots, N - 1$ ($\chi^{(1)}$ is the Perron-Froneius mode) is the orthogonal set of eigenvectors of the Cartan matrix $C_{ln} = 2\delta_{ln} - \delta_{l,n+1} - \delta_{l+1,n}$. Then the density ρ obeys the equation

$$\frac{1}{N}\int_{-B}^{B} R^{(1)}(\theta - \theta^1)\rho(\theta')\,d\theta' = \frac{m}{2\pi}\cosh\theta. \tag{10}$$

Further simplifications occur in the large-N limit,

$$R^{(1)}(\omega) \approx \frac{2N}{\omega}\frac{|\omega|}{\omega^2 + 1}. \tag{11}$$

Now the density ρ may be found in a closed form. To see this, let us apply the operator $(-\partial^2/\partial\theta^2 + 1)$ on both sides of the equation. As a result we obtain an integral equation with the Cauchy kernel $(\theta - \theta')^{-2}$. This equation is solvable,

$$\rho(\theta) = \frac{m}{4K_0(B)\sqrt{B^2 - \theta^2}}, \tag{12}$$

where $K_0(B)$ is the Bessel function. Note that in the large-N limit $R^{(p)}(\omega)$ vanishes at large ω, whereas at finite N it approaches 1. This implies a singular behavior of $\rho(\theta)$ at the Fermi point $\pm B$. As a result the physics on the threshold $h \sim m$ will be changed drastically.

The value of the Fermi rapidities as a function of number of particles can now be obtained from

$$n = \int_{-B}^{B} \rho(\theta)\,d\theta = \frac{\pi m}{4K_0(B)}. \tag{13}$$

In turn the energy of the state with a given number of particles is

$$E/N^2 = \frac{m}{2\pi^2}\int_{-B}^{B}\cosh\theta\,\rho(\theta)\,d\theta = \frac{m^2}{8\pi}\frac{I_0(B)}{K_0(B)}. \tag{14}$$

The field $h = (2\pi^2/N^2)dE/dn$ which corresponds to a given number of particles and the energy as a function of the field $\mathcal{E} = E - \sum_l h_l n_l = E - (N^2/2\pi^2)hn$ are given by

$$\frac{m}{h} = BK_1(B), \tag{15}$$

$$\frac{\mathcal{E}(h)}{N^2} = -\frac{m^2}{8\pi}\frac{I_1(B)}{K_1(B)} \tag{16}$$

Singular behavior on threshold.—The large-N limit exhibits new features on the threshold $h \to m$ ($B \to 0$). At small B we have $I_1(B) \to (B/2 + B^3/16 + \cdots)$ and $K_1(B) \to 1/B + (B/2)\ln(B/2) + \cdots$. Then, from (15) we obtain $B^2 \simeq 4(h/m - 1)|\ln(h/m - 1)|$. This gives singular behavior on the threshold,

$$N^{-2}\mathcal{E}(h) \simeq -\frac{m^2}{4\pi}(h/m - 1)/|\ln(h/m - 1)|. \tag{17}$$

It differs drastically from the threshold behavior for a finite-N theory of massive particles, where we would have (see, e.g., [2]) $\mathcal{E}_N(h) \sim -m^2(h/m - 1)^{3/2}$.

The deviation of the threshold behavior from $\frac{3}{2}$ power law indicates that at $N = \infty$ the theory does not describe any particles, i.e., pointlike objects with asymptotic states. The reason for the singular behavior is the emergence of an extra dimension in the large-N limit—the masses of physical particles are separated from each other by a spacing of order $1/N$, which is less than any energy scale left in the system. Therefore, any external field excites a bundle of particles (an extended object), which is characterized by an extra "momentum" in addition to the usual momentum.

Perturbative regime: Gell-Mann–Low function to all orders.—The large h/m regime is described by perturbation theory. It corresponds to $B \to \infty$. The renormalization properties of the theory are encoded by two Gell-Mann–Low equations: One is for a running (renormalized) coupling constant $h(\partial/\partial h)\bar{\lambda}(h) \equiv \beta(\bar{\lambda})$. The second corresponds to a physical quantity of interest, like the free energy: $h(\partial/\partial h)\ln\mathcal{E}(h) \equiv 2 + \gamma(\bar{\lambda})$. Let us define the running coupling constant as $\bar{\lambda}(h) \equiv 4\pi/B$. At $h \to \infty$ the running constant $\bar{\lambda}(h)$ tends to its bare value λ_0. Then, from (15) we find the result valid to any order of the renormalized perturbation theory

$$\beta(\bar{\lambda}) = -\frac{1}{4\pi}\bar{\lambda}^2\frac{K_1(4\pi/\bar{\lambda})}{K_0(4\pi/\bar{\lambda})}$$
$$= -\frac{1}{4\pi}\bar{\lambda}^2\sum_{n=0}^{\infty} b_n\left(\frac{\bar{\lambda}}{32\pi}\right)^n. \tag{18}$$

$$\frac{\mathcal{E}(h)}{N^2h^2} = -\frac{1}{4\bar{\lambda}}\left[1 - \sum_{n=1}^{\infty}\frac{2n+1}{2n-1}\frac{[(2n)!]^3}{(n!)^4 8^{2n}} \times (\bar{\lambda}/4\pi)^{2n}\right]. \tag{19}$$

Note the asymptotic behavior of the large order coefficients $b_n \sim -(-1)^n \sqrt{8/(\pi n)}\,(4n/e)^n$. In the two-loop approximation this gives

$$-\frac{16\pi \mathcal{E}(h)}{N^2 h^2} = \ln\frac{h}{m} + \frac{1}{2}\ln\ln\frac{h}{m} + \frac{1}{2}\ln\frac{\pi}{2} + \cdots . \tag{20}$$

This result reproduces correctly the one- and two-loop terms of the perturbation theory including the universal constant $\frac{1}{2}\ln(\pi/2)$ (the first calculation of a similar constant was given in [7] for the Kondo problem; for the chiral field models it has been computed in [5,8].).

Despite the fact that every coefficient represents a sum over renormalized planar graphs, it grows factorially with the order. This happens because of the presence of renormalons (subsequences of logarithmically divergent graphs) giving the main factorial contribution to each order (as noticed along time ago by 't Hooft [9]). This means that we have an exponential number of graphs in each order but some of them give $n!$ contributions after the momenta integration.

Another property apparently inherent to any asymptotically free field theory is that all coefficients in (19) are positive rendering the series non-Borel summable. Nevertheless, there is a prescription based on analytical continuation of the Borel transformation for summing up the series (19) to restore the result (16):

$$\frac{\mathcal{E}(h)}{N^2 h^2} = -\frac{\pi}{\bar\lambda^2}\mathrm{Re}\int_0^\infty dt\, e^{-4t\pi/\bar\lambda} F(-1/2, 3/2, 1, t^2/4), \tag{21}$$

where F is the hypergeometric function for the 4D gauge theories. Note that the integral (21), taken along the real axis, possesses also an exponentially small (for $\bar\lambda \to \infty$) imaginary part equal to $-K_1^2(B)B^2/\pi$. The prescription is based on analytical continuation of the Borel transformation and is also valid for some 4D gauge theories [10]. In conclusion, we present an example of an exactly solvable $N = \infty$ matrix model in $1+1$ dimensions. It is conceivable that it describes a string theory in two physical dimensions due to the analogy between planar Feynman graphs and the world sheets of a string. Of course one should not take this analogy literally: In asymptotically free theory neither the coupling constant λ_0 nor a renormalized coupling is a cosmological constant of a string; due to renormalons some small portion of the graphs has a factorially big weight (with respect to the order). However, a less naive string interpretation could be possible. There might exist a (nonperturbative) parameter which controls the size of the graphs in a new perturbation theory. The logarithmic behavior on the threshold reminds us of a similar result for the 1D bosonic string (emerging from the matrix quantum mechanics) [11], if we take $|h - m|$ instead of the cosmological constant $|\lambda - \lambda_{\rm crit}|$. In the 1D matrix model the cosmological constant controls the behavior of the fermions on the top of the Fermi sea and thus it also controls the size of the graphs. The chemical potential $|h - m|$ in the 2D case may conceivably play the same role.

An anticipated feature of the model (also observed in the $c = 1$ matrix model) is the emergence of an extra dimension following from the matrix structure of the theories. This dimension is related to the random walk along the A_{N-1} Dynkin diagram. The structure of this extra (third) dimension is suggested by the fact that the kernel (9) of the spectral equation (10) looks like a propagator for periodic motion in the space of rapidities and Dynkin diagram.

Perhaps the simple result of this paper may be obtained by less sophisticated and more straightforward methods.

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