PHYSICAL REVIEW LETTERS

VOLUME 73 | 26 SEPTEMBER 1994 | NUMBER 13

Nonlinear Dispersion and Compact Structures

Philip Rosenau
Technion, Haifa 32000, Israel
and Center For Nonlinear Studies, MS-B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 17 November 1993)

Relaxing the distinguished ordering underlying the derivation of soliton supporting equations leads to new equations endowed with nonlinear dispersion crucial for the formation and coexistence of compactons, solitons with a compact support, and conventional solitons. Vibrations of the anharmonic mass-spring chain lead to a new Boussinesq equation admitting compactons and compact breathers. The model equation $u_t + [\delta u + 3\gamma u^2/2 + u^{1-\omega}(u^\omega u_x)_x]_x + \nu u_{txx} = 0$ ($\omega, \nu, \delta, \gamma$ const) admits compactons and for $2\omega = \nu\gamma = 1$ has a bi-Hamiltonian structure. The infinite sequence of commuting flows generates an integrable, compacton's supporting variant of the Harry Dym equation.

PACS numbers: 03.40.Kf, 47.20.Ky, 52.35.Sb, 63.20.Ry

In this Letter I derive and study certain nonlinearly dispersive partial differential equations which give rise to both compact and conventional noncompact soliton structures. Observed patterns in nature whether stationary or propagating are usually of finite extent, yet all known conventional solitons though localized are of infinite extent. This may be due to the shortcomings of a continuum theory but, as we shall see, at least in part this is a result of an inadequate mathematical modeling of physical phenomena. A typical derivation of model equations employs a distinguished scaling. In such ordering one parameter controls the balance between a weak nonlinearity and dispersion. (As a by-product dispersion enters only on the linear level.) Unfortunately, this eliminates other possible interplays. In the case of a dense chain presented later, two *independent* small parameters are involved; the anharmonicity of the springs and the equilibrium distance between the mass points. One brings in the nonlinearity, the other induces the dispersion. A definite order between these parameters leads to the Korteweg-de Vries (KdV) equation. But, if we do not tie these parameters in a strict ordering, quadratic effects in dispersion have to be retained. This leads to a new equation and to new effects. This situation is typical, e.g., consider the plasma ion-acoustic waves—if the ion-electron charge separation is tied to the ion's inertia KdV follows. But if these effects are not tied down in a definite ordering then, again, quadratic dispersion has to append the linear one. The equations thus obtained exhibit a variety of conventional and compact support soliton structures.

A nonlinear dispersion model.—Consider

$$Kq(m,\omega):\quad u_t + (u^m)_x + [q(\omega)]_x = 0, \qquad m > 0, \tag{1}$$

where $q(\omega) = u^{1-\omega}(u^\omega u_x)_x$. The $Kq(m,\omega)$ is a nonlinear extension of the KdV. Various values of ω arise in different physical settings. Alternatively, consider the differential-difference equation

$$\dot{u}_j(t) + A(u_j)(u_{j+1} - u_{j-1})/2h = 0.$$

Taking $A(u_j) = u_j$, $(u_{j+1} + u_j + u_{j-1})/3$, or $(u_{j+1} + u_{j-1})/2$ and expanding up to $O(h^4)$ leads to a rescaled form of Eq. (1) with $m = 2$ and $2\omega = -1$, 1, and 2, respectively.

Unlike conventional solitons, the solitary wave solutions of Eq. (1) have a *compact support.* That is, they vanish identically outside a finite domain. In fact, for $m = 2$, and $\omega = 2$, Eq. (1) is just the $K(2,2)$ equation studied in some detail in Ref. [1], where I have introduced the model equation

$$K(m,n):\quad u_t + (u^m)_x + (u^n)_{xxx} = 0. \tag{2}$$

It is the remarkable nature of the $K(m,n)$ compact solutions of Eq. (2), which I call *the compactons,* that motivates the present study. I intend to demonstrate that *compactons are not esoteric mathematical objects* but arise in a wide variety of settings where nonlinear dispersion arises naturally. Though a number of cases with an infinite number of conservation laws was found—the underlying nonlinear mechanism responsible for the coherence and robustness of interaction remains very much a mystery.

Let me recall that though for various members of the $K(m,n)$ family only a finite number of local conservation laws were found, nevertheless collision between the $K(2,2)$ compactons appears to be as elastic as numerical experiments are capable of detecting and they always reemerge with exactly the same coherent shape (see [1] for visualization of this process). Numerically, the $K(2,2)$ always decomposes compact initial data into a number of compactons and possibly anticompacton(s) [1]. These compactons appear to play the role of nonlinear local basis functions. Similar properties are shared by many other equations in the $K(m,n)$ family simulated in [2].

Equation (1) can be rewritten as

$$(u^{\sigma})_t + a(u^{\sigma+m-1})_x + \sigma(u^{\sigma}u_{xx})_x = 0, \qquad (3)$$

where $\sigma = 2\omega + 1$ and $a = m\sigma/(m + 2\omega)$. Hence, both u and u^{σ} are conserved quantities. The availability of other conserved quantities depends on the particular value of ω. For $\omega = \frac{1}{2}$, Eq. (1) is obtained from a Lagrangian density [3], $u = \psi_x$,

$$L = \psi_t\psi_x/2 + (\psi_x)^{m+1}/(m+1) + (\psi_x^2)\psi_{xxx}/4, \qquad (4)$$

with a third conserved quantity $u^{m+1}/(m+1) - uu_x^2/2$. For $\omega = 1$, both $u\cos(x)$ and $u\sin(x)$ are also conserved [1].

From (3) a one parameter family of traveling waves is obtained via the periodic solution of

$$V^{\sigma}(aV^{m-1} + \sigma V_{\zeta\zeta} - 1) = P_0, \qquad (5)$$

where $u = \lambda^{\beta}V(\lambda^{\kappa}\zeta)$, $\beta = 1/(m-1)$, $\kappa = \beta(m-2)/2$, and $\zeta = x - \lambda t$. In particular, for $P_0 = 0$ all $Kq(m = 2, \omega)$ compactons are obtained from the periodic solutions ($C =$ const)

$$u = \lambda s^2[1 + C\cos(\zeta/s)]/\sigma, \qquad s^2 = \omega + 1.$$

As $C \to 1$ these waves turn into a train of compactons separated by a singularity at $u = 0$ where the compactons do not communicate with each other and thus can be split into separate entities given as ($U = 2\lambda s^2/\sigma$)

$$u_c(x,t) = U\cos^2[(x - \lambda t)/2s], \qquad |x - \lambda t| \le s\pi, \qquad (6)$$

and vanish elsewhere. Note that unlike the KdV soliton, which narrows as the amplitude (speed) increases, the invariance of (1) under $u \to \alpha u$ and $t \to t/\alpha$, for $m = 2$ and $\alpha =$ const, implies that compactons width is fixed and independent of the speed. This means that when $m = 2$, there is a detailed balance between nonlinear convection and nonlinear dispersion. Using $\alpha = -1$ also implies negative anticompactons propagating to the right. When $m > 2$, nonlinearity overcomes dispersion and at high amplitudes narrows the resulting compacton.

A glimpse into higher dimensions is given via a nonlinear extension of the Kadomtsev-Petviashvili equation

$$[u_t + (u^2)_x + q(\omega)_x]_x + u_{yy} + u_{zz} = 0, \qquad (7)$$

which can be mapped into Eq. (1) [3]. This results in compactons residing on a traveling paraboloid strip [4].

A mixed dispersion model.—The KdV and the $Kq(m,\omega)$ [or the $K(m,n)$, Eq. (1) or (2)] represent two conceptual extremes in which either linear or nonlinear dispersion is present. However, in physical applications *often* both linear and nonlinear dispersions are present. As different dispersion expressions are usually equivalent, up to asymptotically higher orders, I will use this freedom and seek to append Eq. (1) with linear dispersion which preserves the scaling symmetry of $Kq(m,\omega)$. For $m = 3$ the scaling balance is preserved if Eq. (1) is appended with $\partial_x^5 u$ [5]

$$u_t + (u^3)_x + [q(\omega)]_x + \eta^{\partial_x^5 u} = 0, \qquad (8)$$

and is known to be integrable in three cases [6].

I shall pursue the $m = 2$ case. To preserve the scaling of Eq. (1), it has to be appended with linear dispersion of the form $\partial_x^2 u_t$ and/or $\partial_x^4 u_t$. (The quintic term replaces the cubic term upon its vanishing [7].) Thus Eq. (1) appended with u_{xxt} and convection reads (ν, δ, γ are parameters)

$$Du_t + \delta u_x + 3\gamma uu_x + [q(\omega)]_x = 0, \qquad D = 1 + \nu\partial_x^2. \qquad (9)$$

Equation (9) has the same scaling properties as Eq. (1) and in two cases is found to be integrable in the conventional sense.

I limit myself here to $\omega = 0.5$. To find traveling waves take $\zeta = x - \lambda t$. After two integrations of (9) we have

$$(\delta - \lambda)u^2 + \gamma u^3 + (u - \lambda\nu)u_{\zeta}^2 = P_0u + P_1, \qquad (10)$$

with P_0 and P_1 as constants. If $P_0 = P_1 = 0$, the conventional solitonic pattern follows from ($u = \gamma V$, $\Omega = \nu\gamma$)

$$V_{\zeta}^2 = \gamma V^2(\delta - \lambda + V)/(\lambda\Omega - V). \qquad (11)$$

When $\gamma > 0$, the soliton is a wave of depression for $V > 0$ and $0 < \lambda < \delta$ and an elevation wave for $V < 0$ and $\lambda > \delta$. When $\gamma < 0$ and $V < 0$, the soliton's speed is constrained by $\lambda(1 - \gamma\nu) \le \delta$. In the exceptional case of equality

$$u = \lambda\nu\exp(-\mu|\zeta|), \qquad \mu^2 = -\gamma, \qquad (12)$$

and $\lambda = \delta/(1 - \nu\gamma)$ is a unique speed of propagation. But, if $\nu\gamma = 1$ and $\delta = 0$, every speed λ in (12) becomes admissible.

If $\gamma > 0$ then, in addition to solitons, the system supports compact structures. But unlike the typical soliton case, some compactons emerge for *nonzero values of integration constants.* To unfold this case I rescale Eq. (9) as $x^* = x\mu$, $t^* = t\mu^3$, $\delta^* = \delta/\mu^2$, and $\mu^2 = \gamma$. In the rescaled Eq. (10), $\gamma \to 1$, $\nu \to \nu\gamma = \Omega$, and I combine P_0, and P_1 into one parameter—r. Let $\Sigma = \delta/\gamma\lambda$ and $V = u/\lambda$, then (10) reads

$$(\Omega - V)\left[V^2 + (\Omega + \Sigma - 1)V - r^2 + V_\zeta^2\right] = 0. \tag{13}$$

Equation (13) has an r family of trigonometric solutions

$$V = \sqrt{r^2 + V_0^2}\cos(\zeta) - V_0, \qquad 2V_0 = \Omega + \Sigma - 1. \tag{14}$$

For $r = 0$, a compacton solution emerges

$$V = 2V_0\cos^2(\zeta/2), \qquad |\zeta| \le \pi, \tag{15}$$

and zero elsewhere. In the original variables

$$u = [\lambda(1 - \gamma\nu) - \delta]\cos^2[\sqrt{\gamma}\,(x - \lambda t)/2]/\gamma. \tag{16}$$

For $\gamma = \frac{2}{3}, \delta = \nu = 0$ this is compacton (6). Linear dispersion modifies its speed-amplitude relations. When $V_0 = 0$, (15) vanishes and, if $r = \Omega$, a new compacton $[\lambda = \delta/(1 - \nu\gamma)]$,

$$u = \lambda\nu\cos(x - \lambda t), \qquad |x - \lambda t| \le \pi/2, \tag{17}$$

and zero elsewhere, is born. As in (12), when $\Omega = \nu\gamma = 1$ and $\delta = 0$, every speed λ is admissible. Thus with the collapse of the compacton (16) a new one (17), not possible in Eq. (1), emerges. For this compacton the presence of linear dispersion in a particular balance with convection (i.e., $\nu\gamma = 1$) is essential.

The two exceptional solutions (12) and (17) are mirrors of each other. In both cases $\Omega = \gamma\nu = 1$ and equation (9) may be casted into a bi-Hamiltonian form and thus has an infinite number of conservation laws. For $\gamma > 0$ this will be shown below. For $\gamma = \nu = -1$ the integrability was shown in [8] but holds for all values of $\gamma < 0$ and ν tied by $\Omega = 1$.

To study the exceptional case, I take $\delta = 0$ and $\gamma > 0$. Equation (9) now admits only compactons. In rescaled form

$$\rho_t + 3uu_x + [q(0.5)]_x = 0, \tag{18a}$$

$$\rho = D_+u, \tag{18b}$$

or if $\phi^2 = \rho$ we obtain

$$\phi_t + (u\phi)_x = 0, \tag{19a}$$

$$D_+u = \phi^2, \tag{19b}$$

$$D_\pm = 1 \pm \partial_x^2, \tag{19c}$$

From Eqs. (18) and (19) observe the conservation of ρ and ϕ. By multiplying (18a) by u and integrating by parts, we obtain the conservation of $H_1 = \int u\rho\,dx$. Modifying the Lagrangian (4) yields the Hamiltonian density (energy integral)

$$H_2 = \int u(u^2 - u_x^2)\,dx/2, \tag{20}$$

H_2 and H_1 form a Hamiltonian pair. Therefore, Eqs. (18a) and (18b) may be written in two equivalent forms

$$\rho_t = -\Phi_1\delta H_2/\delta\rho = -\Phi_2\delta H_1/\delta\rho$$

and two Poisson structures defined via

$$\Phi_1 = D_+\partial_x \quad \text{and} \quad \Phi_2 = \partial_x\rho + \rho\partial_x.$$

Combination of operators Φ_1 *and* Φ_2 is exactly of the form used in the KdV theory. Their compatibility assures the bi-Hamiltonian nature of the problem. The recursion operator $\mathcal{R} = \Phi_2\Phi_1^{-1}$ and its transpose are then recursively used to deduce an infinite number of conservation laws [9,10] and hierarchy of commuting flows. From these conservation laws both an upward and downward chain of conserved quantities are formed via recursive relations. For H_{-n}, $n > 0$,

$$\delta H_{-n}/\delta\rho = \Phi_2^{-1}\Phi_1\delta H_{-n+1}/\delta\rho,$$

and all H_{-n} are distinguished by being described in terms of ρ. The same is true for the hierarchy of commuting flows. In particular, the flow generated for $n = 2$ is

$$\rho_t = -\partial_x D_+(\rho^{-0.5})/2. \tag{21a}$$

Define $r = 1/\rho$; then the resulting equation

$$r_t = r^2\partial_x D_+(r^{0.5})/2 \tag{21b}$$

is a *new,* integrable, compacton generating equation. It has *both traveling and stationary compactons*

$$r = \{2\sqrt{\lambda}\cos[3(x + \lambda t)/2]\}^{4/3},$$

and

$$r = r_0\cos^4(x/2)$$

respectively. This may be looked upon as a "compactified" extension of the Harry Dym (HD) equation [11].

Alternatively, note that $(t,x) \to (it, ix)$ transforms one integrable case into the other ($D_+ \to D_-$, $\gamma > 0 \to \gamma < 0$) and thus affords an immediate adaptation of the conserved quantities and flows derived in [8]. Similarly, $D_+ \to D_-$ in (21a) yields a new integrable extension of the HD equation [8]. Again, one begets compactons, the other solitons. While solitons and compactons are different entities, from the point of view of conservation structures these differences are irrelevant. Dual relations between solitons and compactons are studied in more detail elsewhere [11].

For $\delta \ne 0$, one proceeds as before with only small modifications. For $\gamma < 0$, (11) yields smooth solitons. For $\gamma > 0$, it is seen from (16) that conventional solitons are possible, while compactons have a unique amplitude δ. It is unclear whether compact and noncompact solitons emerge simultaneously.

Vibrations of a dense chain.—Consider the motion of N initially equally spaced ($h \ll 1$) mass points m. The potential part of the Hamiltonian is assumed to be

$$H_p = \sum P(S), \qquad S = (y_{n+1} - y_n)/h. \tag{22}$$

$P(S)$ is either a purely anharmonic $P_N(S) = \alpha_N S^N/N$ or a mixed potential $P(S) = \alpha_2 S^2/2 + \alpha_3 S^3/3$. The anharmonicity parameter α_3 is assumed to be numerically small but otherwise is free and is not tied to h in any particular ordering. Expanding in h up to $O(h^4)$, I obtain

$$P_N(S) \to P_N(y_x) = y_x^{N-2}(y_x^2 - h^2 C_N y_{xx}^2), \quad N = 1, 2, 3, \ldots, \tag{23}$$

$$H_p \to H = \int P_N(y_x)\, dx, \qquad C_N = 1/12, 1/4, 1/2, \ldots.$$

and the resulting equation of motion is ($u = y_x, \varepsilon = h^2/12$),

$$u_{tt} = (\alpha_2 u + \alpha_3 u^2)_{xx} + \varepsilon \alpha_2 \partial_x^4 u + 2\varepsilon\alpha_3[q(1/2)]_{xx}. \tag{24a}$$

Equation (24a) admits *both* compactons and conventional solitons [11]. Similarly, for the purely anharmonic, say quartic, potential, in normalized units I obtain

$$u_{tt} = (u^3)_{xx} + [u(u^2)_{xx}]_{xx}, \tag{25}$$

with a purely *cubic* nonlinear dispersion. In addition to the compacton solution $\sqrt{2}\lambda\cos(x - \lambda t)$, Eq. (25) also supports *breathers* of the form $u = Q(t)Z(x)$, where $Q(t)$ satisfies

$$Q''(t) + \kappa^2 Q^3(t) = 0 \tag{26a}$$

(κ is a separation constant) with the periodic solution $Q(t) = cn(\kappa t, 1/\sqrt{2})$, and $Z(x)$ satisfies

$$[Z(Z^2)_{xx}]_{xx} + (Z^3)_{xx} + \kappa^2 Z = 0 \tag{26b}$$

having among others, the following *compact* solution

$$Z = \sqrt{8}\kappa\cos(x/2) \quad \text{when } |x| \le \pi \tag{27}$$

and vanishes elsewhere. While robustness of this particular solution is as yet unknown, extensive numerical studies [1,2] indicate that compactons smoothness at the edge is not indicative of their stability [12].

Equation (24a) is the Boussinesq equation due to Kruskal and Zabusky, but appended with a nonlinear dispersion. Like its predecessor its dispersion has a "wrong" sign. One can synthesize from (24a) a one-sided equation or regularize to underlying expansion [7]. To this end the discrete energy (22) is replaced with a Pade approximant, which unlike $y_x^2 - \varepsilon y_{xx}^2$, derived by Taylor expansion, preserves the boundedness of the original potential energy. Thus $(y_x \mathbf{L} y_x)$, where in Fourier space $\hat{\mathbf{L}}(k) = 4\sin^2(kh/2)$, is approximated by a bounded operator $\hat{\mathbf{L}}_A = 1/(1 + \varepsilon k^2)$. Explicitly,

$$H_p = \sum P_2(S) \to \int (y_x \mathbf{L} y_x)\, dx,$$

with the resulting equation of motion

$$u_{tt} = (\alpha_2 u + \alpha_3 u^2)_{xx} + \varepsilon\partial_x^2 u_{tt} + 2\varepsilon\alpha_3[q(1/2)]_{xx}. \tag{24b}$$

Equation (24b) unlike (24a) is well posed. It admits both compact and conventional solitons. The model Eq. (9) is its one-sided version. A similar regularization of (25) appends it with εu_{xxtt} and results in a compact breather similar to (26a).

In summary, the equation governing the motion of a dense chain is a prototype of compacton supporting equations. The derivation relaxes the distinguished scaling and is also applied to study the motion of ion-acoustic waves [11] and a flow of a two-layer liquid [13]. It yields compacton supporting equations. Among the solutions of the prototypical Eq. (9), two *integrable* cases were found.

I thank Dr. Camassa for his informative remarks. This work was supported in part by the AFOSR under Grant No. F49620 J-0054 and in part through the Fund for Promotion of Research at the Technion.

[1] P. Rosenau and J.M. Hyman, Phys. Rev. Lett. **70**, 564 (1993).

[2] J.M. Hyman and P. Rosenau (unpublished).

[3] P. Rosenau, in *Nonlinear Coherent Structures in Physics and Biology*, edited by K.H. Spatschek, and F.G. Mertens, NATO ARW (Plenum, London, 1994).

[4] To map Eq. (7), define $\tau = \ln(1 + t)$, $\xi = x + (y^2 + z^2)/4(1+t)$, and $u = V(\tau, \xi)/(1 + t)$. Then after one integration I obtain Eq. (1) in $V(\tau, \xi)$. Using (6) in $V(s = \xi - \lambda\tau)$ yields a solution of a decaying amplitude in a strip, $|s| \le q\pi$, contained between two evolutionary paraboloids that open up as they traverse to the right and at infinity rectify into a straight strip. It resembles wave crests on a thin sheet of water. Such semicompact structures are perhaps also related to the horseshoe solitons observed to form on a vertically flowing film of water glycerin mixture (cf. V.I. Petviashvili and O.Y. Tsvelodub, Sov. Phys. Dokl. **231**, 17 1978).

[5] Notably, no integrable quintic KdV with an *independent* cubic dispersion [i.e., not obtained by rescaling and shifts of variables in Eq. (9)] was ever derived—interaction of these dispersion mechanisms with convection generates two contradicting scales. In integrable equations I am aware of, there is only one interaction scale between dispersion and convection.

[6] S. Kichenassamy and P.J. Olver, Siam J. Math. Anal. **23**, 1141 (1992).

[7] P. Rosenau, Prog. Theor. Phys. **79**, 1028 (1988); P. Rosenau, Physica (Amsterdam) **27D**, 224 (1987); Phys. Rev. B **36**, 5868 (1987); Phys. Rev. A **39**, 6614 (1989).

[8] R. Camassa and D. Holm, Phys. Rev. Lett. **71**, 1661 (1993).

[9] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1986).

[10] The bi-Hamiltonian structure is a way to explore new integrable systems. A consistent redefinition of ρ and H_2 generates a candidate for a new pair. Thus if $\rho = u_{xx}$ and $H_2 = \int u u_x^2\, dx$ then after one integration another integrable system $(u_t + u u_x)_x = u_x^2/2$ follows [compare with (18b)].

[11] P. Rosenau (unpublished).

[12] The numerical robustness of the various $K(m,n)$ compactons studied in [1,2] does not seem to be affected by, or related to, the smoothness of u at the edge. Thus while for the $K(m = 2,3; n = 3)$ compactons $u = [37.5 - \zeta^2]/30$, $\pm\sqrt{3\lambda/2}\cos(\zeta/3)$, $\xi = x - \lambda t$, the first derivative is discontinuous at the edge (so that the compacton solution must be understood in the weak sense) their stability does not appear to be different from the stability of the $K(2,2)$ compacton [cf. (6)] which is a strong solution of (2) (u^2 has three smooth derivatives). Note also the limited smoothness of the compacton in the *integrable* HD Eq. (21b). Similar analytical and numerical results were obtained for the peaked soliton (12) studied in [8]. Of course, the mere existence of a formally compact structure does not imply robustness. In fact, let $u_{tt} - u_{xx} = u - u^{1/2}$, $u = 16\cos^4[\zeta/4\sqrt{1-\lambda^2}]/9$, where $|\zeta| \leq 2\pi\sqrt{1-\lambda^2}$. Here even four derivatives of u are of no help. Numerical experiments show that the low order dispersion is unable to stabilize the compacton which decomposes immediately into a train of waves.

[13] K.W. Chow (unpublished).