Dynamics by White-Noise Hamiltonians

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A new class of random quantum-dynamical systems in continuous space is introduced. Each member of the class is characterized by a Hamiltonian which is the sum of two parts. While one part is deterministic, time independent, and quadratic, the Weyl-Wigner symbol of the other part is a homogeneous Gaussian random field which is δ correlated in time and arbitrary, but smooth in position and momentum. Exact expressions for the time evolution of both averaged states and observables are obtained. If the deterministic part is that of a particle subject to a constant magnetic field, spatial variance of the averaged state grows diffusively for long times independent of the initial state.

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The spatial spreading of a state under the free time evolution is a well-known and fundamental phenomenon in nonrelativistic quantum and classical mechanics [1]. In order to make a quantitative statement, let σ_t^2 denote the variance of the position at time t of a spinless point particle moving in continuous space. It is assumed that the particle was prepared initially in some state, which is normalized but not necessarily pure. For a free particle a simple calculation then shows that σ_t^2 increases asymptotically for large t as $\sigma_t^2 \sim t^2\varrho_0^2/m^2$. Here $m > 0$ is the mass of the particle and ϱ_0^2 is the variance of its momentum in the initial state. This relation holds both in the quantum and in the classical case. However, classical states, e.g., pure ones, may have a sharp momentum, that is, $\varrho_0^2 = 0$, whereas $\varrho_0^2 > 0$ for all quantum states, including the pure ones ("wave-packet spreading"). It is clear that σ_t^2 can also be calculated exactly for a time evolution governed by a more general Hamiltonian being at most quadratic in momentum and position [2].

A variety of physical systems, whose properties are unknown in detail, are successfully modeled by a Hamiltonian with a random part added to a (simple) deterministic part. Thus it is a challenging problem to study the properties of the random variable σ_t^2 and of related key quantities in these models. For recent investigations devoted to the question, how the ballistic long-time behavior of σ_t^2 of the free particle is modified by adding a Gaussian time-dependent random potential, see [3–6] and references therein. A stepping stone in this direction was the work of Jayannavar and Kumar [3], who—in building on treatments of lattice models [7]—exploited the simplifying feature of a vanishing correlation time ("Gaussian white-noise potential"). Their main result concerns the quantum case with a particular pure initial state. They derived an exact expression for the spatial variance Σ_t^2 of the averaged state at time t and found $\Sigma_t^2 \sim t^3$ for large t. (Note that the averaged spatial variance $\overline{\sigma_t^2}$ never exceeds Σ_t^2.) Interesting attempts to incorporate a nonzero correlation time by the use of perturbative methods—with partially conflicting results—can be found in [5].

However, several problems of considerable interest have not been tackled so far or deserve further study. First, in order to describe the effects of externally applied force fields, one must not restrict the deterministic part of the Hamiltonian to that of a free particle. For example, the presence of an electric field is discussed in [8]. Here we will see that a constant magnetic field leads to a diffusive behavior in the sense that $\Sigma_t^2 \sim t$, a result with some relevance for magnetotransport theory. Second, the random part can be generalized to cover the case of a momentum-dependent (in other words, nonlocal) random potential, which is the continuous-system analog of off-diagonal disorder in lattice systems [7]. This is of interest, for example, to caricature the effective motion of a test particle due to inelastic scattering by the irregular motion of other particles. It suggests itself also from the point of view of Hamiltonian mechanics. And last, we will show that the above-mentioned noise-induced results are neither affected by quantum fluctuations nor do they depend on the initial state.

In fact, it is the main purpose of the present Letter to demonstrate that there is a rather general class of Gaussian white-noise Hamiltonians for which one can obtain exact and explicit results on the averaged time evolution. Yet before we describe this class in detail, it seems adequate to comment on the representation we are going to use throughout.

We consider a quantum-mechanical system which, for simplicity, has the Euclidean line $\mathbb{R}$ as its configuration space. The extension to the d-dimensional Euclidean space $\mathbb{R}^d$ is merely a matter of notation. Since the random part of the Hamiltonian will be allowed to depend on both position and momentum, it is convenient to characterize its properties in terms of those of an associated random function on classical phase space $\mathbb{R} \times \mathbb{R}$. Therefore, and in order to treat the classical limit with low effort, it is only consequent to represent the quantum system entirely in phase space. The representation we choose is the one dating back to ideas of Weyl, Wigner, and Moyal [9], where a quantum operator $\hat{f}$ acting in the Hilbert space of

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square-integrable functions on $\mathbb{R}$ is represented uniquely by its symbol, that is, by the phase-space function

$$f(p,q) := \int_{\mathbb{R}} dr\, e^{ipr/\hbar} \langle q - r/2|\hat{f}|q + r/2\rangle . \quad (1)$$

We recall that the symbol of the standardized commutator $(i/\hbar)(\hat{f}\hat{g} - \hat{g}\hat{f})$ of two operators $\hat{f}$ and $\hat{g}$ is the Moyal bracket of their respective symbols

$$[f,g](p,q) := f(p,q) \times \frac{2}{\hbar} \sin\left\{\frac{\hbar}{2}\left(\overleftarrow{\partial}_p\overrightarrow{\partial}_q - \overleftarrow{\partial}_q\overrightarrow{\partial}_p\right)\right\} g(p,q) .$$

For nonpolynomial f and g it is often advantageous to rewrite $[f,g]$ as a Fourier-integral expression. Of course, in the classical limit, when Planck's constant $\hbar$ tends to zero, the Moyal bracket reduces to the Poisson bracket. An observable, corresponding to a self-adjoint operator, has a real symbol a. A quantum state is represented by a Wigner density w which, by definition, is $(2\pi\hbar)^{-1}$ times the symbol of the corresponding "density matrix," that is, of a positive unit-trace operator. The expectation value (or mean) of the observable a in the state w is then given by the scalar product

$$\langle w,a\rangle := \int_{\mathbb{R}\times\mathbb{R}} dp\, dq\, w(p,q)a(p,q) .$$

We recall that $|w(p,q)| \leq (\pi\hbar)^{-1}$ and that $\langle w,1\rangle = 1$. Moreover, $\langle w,w\rangle \leq (2\pi\hbar)^{-1}$ with equality if and only if w represents a pure quantum state. In the classical limit a quantum state w converges (weakly) to a probability density on phase space, that is, to a classical state.

Now we are in a position to introduce a dynamics governed by a Hamiltonian on phase space which we call white-noise Hamiltonian

$$H(p,q) + N(p,q;t) . \quad (2)$$

By definition, the Hamiltonian operator is obtained from (2) by inverting (1), that is, by Weyl ordering. For simplicity, the deterministic part $H(p,q)$ is supposed to be time independent and at most quadratic in p and q. The random part $N(p,q;t)$ is supposed to be a Gaussian white-noise field with mean zero and covariance

$$\overline{N(p,q;t)N(p',q';t')} = C(p - p', q - q')\delta(t - t') . \quad (3)$$

Here the overbar denotes averaging with respect to the probability distribution of N and homogeneity is assumed just for brevity. By its probabilistic origin, the covariance function C may be any even phase-space function with a non-negative (symplectic) Fourier transform

$$\tilde{C}(x,k) := \int_{\mathbb{R}\times\mathbb{R}} \frac{dp\, dq}{(2\pi)^2} C(p,q)\cos(xp - kq) \geq 0 .$$

For later purpose we will assume that C is sufficiently smooth, equivalently, that the probability density $\tilde{C}(x,k)/C(0,0)$ has moments of sufficiently high order.

In the Schrödinger picture the time evolution $w_0 \mapsto w_t$ of a given initial state w_0 is determined by the stochastic quantum Liouville equation [10] associated with (2)

$$\partial_t w_t = [w_t, H] + [w_t, N(t)] . \quad (4)$$

Here the bracket $[w_t, H]$ is in fact a Poisson bracket due to the quadratic nature of H. In order to derive an equation of motion for the averaged state $\overline{w_t}(p,q) := \overline{w_t(p,q)}$ from (4), we follow essentially the earlier treatments in [3,7] and perform a functional integration by parts with respect to the Gaussian average [11]. In doing so, we think of the Dirac delta function in (3) as being approximated by a sequence of smooth covariance functions with correlation time tending to zero, which amounts to the Stratonovich interpretation [12] of (4). The final result can be cast into the form of the linear integro-differential equation

$$\partial_t \overline{w_t}(p,q) = [\overline{w_t}, H](p,q) + \frac{1}{\hbar^2} \int_{\mathbb{R}\times\mathbb{R}} dx\, dk\, \tilde{C}(x,k) \times \{\overline{w_t}(p + \hbar k, q + \hbar x) - \overline{w_t}(p,q)\} \quad (5)$$

which is valid for $t > 0$ and has to be supplemented by the initial condition $\overline{w_0} = w_0$. Several remarks are in order:

(i) Equation (5) is a substantial generalization of the main result of [3]. In the special case of a free deterministic part, $H = p^2/2m$, and a momentum-independent noise, $\tilde{C}(x,k) \propto \delta(x)$, it reduces to an equation which is equivalent to Eq. (8) in [3]. Furthermore, the subsequent treatment of Eq. (8) in [3] is restricted to a pure initial state represented by a joint Gaussian w_0 with $\langle w_0, p\rangle = \langle w_0, q\rangle = \langle w_0, pq\rangle = 0$, $\langle w_0, q^2\rangle = \sigma_0^2 \neq 0$, and $\langle w_0, p^2\rangle = (\hbar/2\sigma_0)^2$.

(ii) The averaged time evolution $\mathcal{T}_t : w_0 \mapsto \overline{w_t}$ given by (5) provides an example of a quantum-dynamical semigroup [13,14] which is monotone mixing increasing. By this we mean that $\mathcal{T}_t$ maps Wigner densities linearly to Wigner densities, $\mathcal{T}_t \circ \mathcal{T}_{t'} = \mathcal{T}_{t+t'}$ for all $t,t' \geq 0$, $\mathcal{T}_0 =$ identity, and $\partial_t \langle \mathcal{T}_t(w_0), \mathcal{T}_t(w_0)\rangle \leq 0$ (with equality only in the uninteresting case where the covariance function C is a constant). The inequality follows from scalar multiplication of (5) by $\overline{w_t} = \mathcal{T}_t(w_0)$, by observing $\langle \overline{w_t}, [\overline{w_t}, H]\rangle = 0$, the Cauchy-Schwarz inequality, and $\tilde{C} \geq 0$. To summarize, the average over randomness has turned the fully reversible quantum Liouville equation (4) into Eq. (5) with coherence-destructing irreversible behavior.

(iii) Interestingly enough, modified quantum-dynamical equations similar to (5) are discussed in very different branches of physics. These include quantum theories not only of certain disordered systems, but also of the measurement process [15], of Markovian transport [13], and of the evaporation of black holes [16].

(iv) As for the generator of the semigroup, one deduces from Eq. (5) that

$$\mathcal{T}_t = e^{-t(\mathcal{L} + \mathcal{N})} \quad (6)$$

with the unperturbed Liouville operator

$$\mathcal{L} := (\partial_p H)\partial_q - (\partial_q H)\partial_p \quad (7)$$

and the noise-induced, irreversibility causing operator

$$\mathcal{N} := \hbar^{-2}\{C(0,0) - C(-i\hbar\partial_q, i\hbar\partial_p)\}.$$

(v) For the derivation of explicit results it is often useful to isolate the unperturbed time evolution in (6) according to standard perturbation theory

$$\mathcal{T}_t = e^{-t\mathcal{L}} \exp\left\{-\int_0^t ds\, e^{s\mathcal{L}}\, \mathcal{N}\, e^{-s\mathcal{L}}\right\}. \quad (8)$$

We note that due to the quadratic nature of H the operators $e^{s\mathcal{L}}\,\mathcal{N}\,e^{-s\mathcal{L}}$ commute at different times s and can be written more explicitly as

$$e^{s\mathcal{L}}\,\mathcal{N}\,e^{-s\mathcal{L}} = \hbar^{-2}\{C(0,0) - C(-i\hbar\mathcal{K}_s, i\hbar\mathcal{X}_s)\}. \quad (9)$$

Here we have introduced the time-dependent first-order differential operators $\mathcal{K}_s := (\partial_q e^{-s\mathcal{L}} q)\partial_q + (\partial_q e^{-s\mathcal{L}} p)\partial_p$ and $\mathcal{X}_s := (\partial_p e^{-s\mathcal{L}} p)\partial_p + (\partial_p e^{-s\mathcal{L}} q)\partial_q$, whose coefficients are obtained from the phase-space trajectory of the unperturbed problem as indicated, and do not depend on p and q.

(vi) By using (8) and (9) it is straightforward to derive a Fourier-integral expression for the integral kernel $\mathsf{T}_t(p,q|p',q')$ of $\mathcal{T}_t$, which is the solution of (5) with initial condition $\mathsf{T}_0(p,q|p',q') = \delta(p-p')\delta(q-q')$. Since this expression is somewhat lengthy and will not be needed below, we omit it.

(vii) The quantum-dynamical semigroup $\mathcal{T}_t$ admits also a purely classical interpretation. This is because it is positivity preserving, which follows from (8) and the fact that $C(-i\hbar\partial_q, i\hbar\partial_p)$ and $e^{\pm s\mathcal{L}}$ are positivity preserving. Therefore, $\mathcal{T}_t$ maps classical states to classical states. In other words, its integral kernel $\mathsf{T}_t(p,q \mid p',q')$ can be interpreted as the transition density of a stationary, in general non-continuous, Markov process in phase space and Eq. (5) may be viewed as the associated classical kinetic equation. In fact, up to the drift arising from H, it is a linear Boltzmann equation with a (homogeneous) stochastic kernel [17] and an increasing Boltzmann-Gibbs entropy: $-\partial_t\langle\mathcal{T}_t(w_0), \ln \mathcal{T}_t(w_0)\rangle \geq 0$. It is used, e.g., in quasiclassical theories of charge transport in semiconductors [18].

(viii) Assuming an $\hbar$-independent covariance function C, one has the expansion

$$\mathcal{N} = -D_{0,2}\partial_p^2 - D_{1,1}\partial_p\partial_q - D_{2,0}\partial_q^2 + \mathcal{O}(\hbar^2\partial^4), \quad (10)$$

where the three constants $D_{0,2}$, $D_{1,1}$, and $D_{2,0}$ as defined through $D_{\mu,\nu} := (-i\partial_p)^\mu(i\partial_q)^\nu C(0,0)/\mu!\,\nu!$ reflect the curvature of C at the origin and obey the inequalities $D_{0,2} \geq 0$ and $4D_{2,0}D_{0,2} \geq D_{1,1}^2$ due to $\tilde{C} \geq 0$. As a consequence, in the classical limit Eq. (5) reduces to a Fokker-Planck type of equation in phase space with drift and diffusion as given by (7) and (10).

Now we return to the problem posed in the beginning of this Letter, namely to evaluate the averaged expectation value $\overline{\langle w_t, a\rangle}$ of a simple observable a at time t, given the initial state w_0. For this purpose it is useful to switch to the Heisenberg picture according to

$$\overline{\langle w_t, a\rangle} = \langle \overline{w_t}, a\rangle = \langle \mathcal{T}_t(w_0), a\rangle =: \langle w_0, \mathcal{T}_t^*(a)\rangle. \quad (11)$$

The thus defined adjoint semigroup $\mathcal{T}_t^*$ can be obtained from (6) or (8) by reversing the sign of $\mathcal{L}$.

To be more specific, we first choose the deterministic part of the Hamiltonian (2) to be that of a free particle, $H = p^2/2m$. Taking the observables p, q, p^2, pq, q^2, and q^4 as examples, one then finds explicitly

$$\mathcal{T}_t^*(p) = p, \quad \mathcal{T}_t^*(q) = q + tp/m, \quad (12a)$$

$$\mathcal{T}_t^*(p^2) = p^2 + 2tD_{0,2}, \quad (12b)$$

$$\mathcal{T}_t^*(pq) = p(q + tp/m) + tD_{1,1} + t^2D_{0,2}/m, \quad (12c)$$

$$\mathcal{T}_t^*(q^2) = (q + tp/m)^2 + 2Q_3(t), \quad (12d)$$

$$\mathcal{T}_t^*(q^4) = (q + tp/m)^4 + 12\{(q + tp/m)^2Q_3(t) + 2\hbar^2Q_5(t) + [Q_3(t)]^2\}. \quad (12e)$$

Here $Q_\mu(t) := \sum_{\nu=1}^{\mu} t^\nu D_{\mu-\nu,\nu-1}\, m^{1-\nu}/\nu$ is a polynomial of (highest) degree μ in time. Quantities such as the spatial variance of the averaged state $\Sigma_t^2 := \langle w_0, \mathcal{T}_t^*(q^2)\rangle - \langle w_0, \mathcal{T}_t^*(q)\rangle^2$ or the averaged mean-square displacement $\Delta_t^2 := \langle w_0, \mathcal{T}_t^*(q^2) - 2q\mathcal{T}_t^*(q) + q^2\rangle$ at time t, may now be obtained immediately.

The exact results (12) illustrate important features valid for general quadratic H: Observables which are linear in p and q are not affected by the white noise N. Moreover, noise-induced terms in averaged expectation values (11) of quadratic observables are independent of the initial state. Assuming an $\hbar$-independent covariance function C, noise-induced effects affected by quantum fluctuations occur only for observables of at least fourth order in p and q. However, very special situations are needed for quantum effects to show up in the leading term for long times. For example, taking the observables p^n or q^n, $4 \leq n$ integer, one must require the phase-space trajectories of H to grow exponentially in time. This remark contradicts certain expressions in [6], since the underlying "correlation functions" considered there do not have a positive Fourier transform, and are therefore physically insignificant.

White-noise Hamiltonians may reveal a diffusive behavior in a weak sense, that is, $\mathcal{T}_t^*(p^2) \sim t$ and/or $\mathcal{T}_t^*(q^2) \sim t$ for long times. The simplest case for weak diffusion to occur in both momentum and position corresponds to $H = 0$, as follows from (12b) and (12d) in the limit $m \longrightarrow \infty$. It occurs also if the deterministic part describes a harmonic oscillator and, more strikingly, in the case of a particle with electric charge $-e$ moving in the Euclidean plane $\mathbb{R}^2$ under the influence of a perpendicular constant magnetic field of strength $|m\omega/e|$. In the latter case we choose $H = (2m)^{-1}\{(p_1 - m\omega q_2/2)^2 + (p_2 + m\omega q_1/2)^2\}$ and, for the sake of brevity, we assume the covariance function $C(\mathbf{p},\mathbf{q})$ to depend only on the lengths of $\mathbf{p} := (p_1, p_2)$ and $\mathbf{q} := (q_1, q_2)$. By a simple extension of the presented methods to higher dimensions, one then finds for the averaged squared position at arbitrary time t the exact result

$$\mathcal{T}_t^*(\mathbf{q}^2) = (e^{t\mathcal{L}}\mathbf{q})^2 - t\left\{\left(1 + \frac{\sin\omega t}{\omega t}\right)\partial_{p_1}^2 C(\mathbf{0},\mathbf{0}) + \left(1 - \frac{\sin\omega t}{\omega t}\right)\left(\frac{2}{m\omega}\right)^2 \partial_{q_1}^2 C(\mathbf{0},\mathbf{0})\right\}. \quad (13)$$

Here $e^{t\mathcal{L}}\mathbf{q}$ is nothing but a cyclotron orbit associated with H. As in lattice models [7], but unlike the free-particle case (12d), the leading term in (13) for large $t \gg \omega^{-1}$ is influenced by the noise in both momentum and position. Note also that the long-time behavior of $\mathcal{T}_t^*(\mathbf{q}^2)$, and hence of Σ_t^2 and Δ_t^2, changes abruptly from a cubic to a linear behavior when turning on a magnetic field. When the fluctuating environment is modeled by colored noise, we expect the same trend. However, the growth should be slower and, as a new feature, it should depend on the initial state. To verify these conjectures, nonperturbative methods are needed for controlling the long-time behavior of quantities as Σ_t^2 and Δ_t^2 in these non-Markovian models.

In the *strictu sensu* continuum model considered here, nonsmooth covariance functions as $C(p,q) \propto \delta(q)$ do not lead to finite results, unless one performs a suitable lattice regularization. In doing so, a diffusive behavior with a saturating averaged energy can be obtained as in [4], even for $H = p^2/2m$.

Returning to smooth covariance functions, it is not surprising that, except for very particular deterministic parts such as $H = (p^2 - Dq^2)/2m$ with $D \geq D_{0,2}/D_{2,0}$, white-noise perturbations lead to a linear increase in time of the averaged energy

$$\mathcal{T}_t^*(H) = H + t(D_{0,2}\partial_p^2 H + D_{1,1}\partial_p\partial_q H + D_{2,0}\partial_q^2 H).$$

To compensate for this effect and, if possible, to allow for an eventual approach to a stationary (equilibrium) state, dissipation has to be incorporated, typically by coupling the white-noise system to a heat bath in the spirit of [19]. In the context of noise and dissipation, Accardi's program "quantum stochastic mechanics" [20] is very promising for a deeper understanding of quantum time evolutions.

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