Exact Dynamical Correlation Functions of Calogero-Sutherland Model and One-Dimensional Fractional Statistics

Z. N. C. Ha

School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540

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One-dimensional model of nonrelativistic particles with inverse-square interaction potential known as the Calogero-Sutherland model is shown to possess fractional statistics. Using the theory of Jack polynomials the exact dynamical density-density correlation function and the hole propagator part of one-particle Green's function at any rational interaction coupling constant are obtained and used to show clear evidences of the fractional *exclusion* statistics in the sense of Haldane's "generalized Pauli exclusion principle." This model is also endowed with the corresponding *natural exchange* statistics.

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The quantum particles obeying fractional statistics known as anyons have been a subject of intense study since the discovery of the fractional quantum Hall effect and high T_c superconductors [1]. The subject, however, is far from complete. In fact, even the simplest model of anyons, namely the ideal gas, has not yet been fully understood. To add to the confusion, there are now two seemingly nonequivalent definitions of fractional statistics. While the popular definition of anyons is based on the quantum phase arising from the exchange of particles [1], Haldane's definition [2] is based on the so-called "generalized Pauli exclusion principle." The main difference between the two approaches is that while in the former the statistics is assigned to the Newtonian point particles, in the latter it is obeyed by the elementary excitations of condensed matter system.

There are no fully solvable two-dimensional models where the ideas of fractional statistics can be rigorously tested. In one dimension, however, there is such a model known as the Calogero-Sutherland model (CSM) [3]. I show in this Letter that the two definitions of fractional statistics can coexist in CSM without any inconsistency. Following Ref. [4], I construct the motifs for all the excited states and explicitly demonstrate that the quasiparticles and quasiholes obey the Haldane's exclusion statistics. I also solve the exact ground state dynamical density-density correlation function (DDDCF) and the hole propagator part of the one-particle Green's function and show that the contributing intermediate states involve only a finite number of quasiparticles (holes) consistent with the ideal "anyon" gas structure of this model. In calculating the correlation function and propagator a new mathematical technique based on the theory of Jack symmetric orthogonal polynomials [5] is used.

The CSM Hamiltonian, which describes a system of N nonrelativistic particles interacting with inverse-square exchange, is given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<l} \frac{2\lambda(\lambda - 1)}{d^2(x_j - x_l)}, \tag{1}$$

where $\hbar^2/2m = 1$ and $d(x_i - y_j)$ is the chord distance between the ith and jth particles on a ring of length L and is equal to $|(L/\pi)\sin[\pi(x_i - y_j)/L]|$. The dimensionless interaction coupling constant λ is a positive real number that specifies the natural statistics of this model. For the special values of $\lambda = 1/2$, 1, and 2, the model is related to the orthogonal, unitary, and simplectic random matrix theory [6], respectively, and DDDCF for those values have previously been found by the supersymmetry technique [7]. For $\lambda = 2$, the full one-particle Green's function has also been found [8,9].

One of the main results of this Letter is the calculation of the dynamical density-density correlation function at any rational interaction parameter $\lambda = p/q$. It is given by

$$\langle 0|\rho(x,t)\rho(0,0)|0\rangle = C\prod_{i=1}^{q}\left(\int_0^{\infty} dx_i\right)\prod_{j=1}^{p}\left(\int_0^{1} dy_j\right) Q^2 F(q,p,\lambda|\{x_i,y_j\})\cos(Qx)e^{-iEt}, \tag{2}$$

where Q and E, the total momentum and energy, are given in units of $\hbar$ and $\hbar^2/2m$ by

$$Q = 2\pi\rho\left(\sum_{j=1}^{q} x_j + \sum_{j=1}^{p} y_j\right), \tag{3}$$

$$E = (2\pi\rho)^2\left(\sum_{j=1}^{q} \epsilon_P(x_j) + \sum_{j=1}^{p} \epsilon_H(y_j)\right), \tag{4}$$

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where $\rho = N/L$, $\epsilon_P(x) = x(x + \lambda)$, and $\epsilon_H(y) = \lambda y(1 - y)$. $x_j(\epsilon_P)$ and $y_j(\epsilon_H)$ are normalized momentum (energy) of the quasiparticles and the quasiholes, respectively. The normalization constant C is given by

$$A(m,n,\lambda) = \frac{\Gamma^m(\lambda)\,\Gamma^n(1/\lambda)}{\prod_{i=1}^m \Gamma^2(p - \lambda(i-1)) \prod_{j=1}^n \Gamma^2(q - (j-1)/\lambda)} \prod_{j=1}^n \left(\frac{\Gamma(q - (j-1)/\lambda)}{\Gamma(1 - (j-1)/\lambda)}\right)^2, \tag{5}$$

$$C = \frac{\lambda^{2p(q-1)}\Gamma^2(p)}{2\pi^2 p! q!} A(q,p,\lambda). \tag{6}$$

Finally, the form factor $F(q,p,\lambda|\{x_i,y_j\})$ is given by

$$F(m,n,\lambda|\{x_i,y_j\}) = \prod_{i=1}^q \prod_{j=1}^p (x_i + \lambda y_j)^{-2} \frac{\left(\prod_{i<j}(x_i - x_j)^2\right)^\lambda \left(\prod_{i<j}(y_i - y_j)^2\right)^{1/\lambda}}{\prod_{i=1}^q \epsilon_P(x_i)^{1-\lambda} \prod_{j=1}^p \epsilon_H(y_j)^{1-1/\lambda}}. \tag{7}$$

The results of Simons *et al.* at $\lambda = 1/2$, 1, and 2 with an appropriate change of variables agree with Eq. (2) up to the normalization constant. Haldane guessed the correct form factor based on the information given by Simons *et al.*, Galilean invariance, and U(1) conformal field theory [10].

The ground state of CSM is given by

$$\Psi_0 = \prod_{j>l}(z_j - z_l)^\lambda \prod_{j=1}^N z_j^{J_0}, \tag{8}$$

where $z_j = \exp(i2\pi x_j/L)$ and $J_0 = -\lambda(N-1)/2$. If a general eigenfunction with energy E is written as $\Psi = \Psi_0\Phi$, then Φ is an eigenstate of the following new effective Hamiltonian H'

$$H' = \sum_i (z_i\partial_{z_i})^2\Phi + \lambda \sum_{i<j} \frac{z_i + z_j}{z_i - z_j}(z_i\partial_{z_i} - z_j\partial_{z_j})\Phi, \tag{9}$$

and satisfies $H'\Phi = \varepsilon\Phi$ where $\varepsilon = (L/2\pi)^2(E - E_0)$. An amazing coincidence happens if $\Delta = (\lambda N - \lambda - 1)\sum_j z_j\partial_{z_j}$ is added to Eq. (9). A complete set of linearly independent solutions of the resulting equation is known in mathematical literature as the Jack polynomial $J_\kappa^{1/\lambda}(z_1,\dots,z_N)$ [5]. The index $\kappa = (\kappa_1,\kappa_2,\dots,\kappa_N)$ is the partition of non-negative integers and is essentially a set of bosonic quantum numbers used to label all the eigenstates of CSM up to global Galilean boosts. In particular, the ground state is given by the partition of zero (i.e., $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$). The parts κ_j's of partition κ are ordered so that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N$. Since the J_κ is a homogeneous symmetric polynomial of degree $|\kappa| = \sum_j \kappa_j$, it is also an eigenfunction of Δ with eigenvalue $(\lambda N - \lambda - 1)|\kappa|$ and thus of H' with $\varepsilon = \sum_j \kappa_j^2 + \lambda(N + 1 - 2j)\kappa_j$. If the pseudomomenta k_j's are defined by

$$Lk_j = 2\pi I_j + \pi(\lambda - 1)\sum_l \mathrm{sgn}(k_j - k_l), \tag{10}$$

with $I_j = \kappa_j + (N + 1 - 2j)/2$, then the eigenenergy and momentum take the following free form: $E = \frac{\hbar^2}{2m}\sum_j k_j^2$ and $P = \hbar\sum_j k_j$. Using a method developed by Yang and Yang [11], Sutherland calculated the thermodynamics for CSM [12] and found that the densities of occupied (ρ_P) and unoccupied (ρ_H) k satisfy the following relation: $\lambda\rho_P(k) + \rho_H(k) = 1$. This is a statement of *broken particle-hole symmetry* for $\lambda \neq 1$ and an analog of the Chern-Simons duality. $\rho_P(k)$ satisfies the following relation [13]: $(1 - \lambda\rho_P)^\lambda[1 - (\lambda - 1)\rho_P]^{\lambda-1} = \rho_P \exp\{[\epsilon(k) - \mu]/T\}$, with $\epsilon(k) = k^2$, which is identical to that of the ideal gas obeying Haldane's fractional exclusion statistics [14]. I emphasize here that the statistical distribution above is only for the pseudoparticles and is not related to the statistics of the real particles.

I use three different names for the particles in the model—real, pseudoparticles, and quasiparticles. The real particles are, of course, the physical quantum particles described by the canonically conjugate coordinate and momentum variables $\{x_j, p_j\}$. The pseudoparticles are described by the pseudomomentum operators [see Eq. (15)] whose eigenvalues are given by Eq. (10). The quasiparticles are the elementary excitations of the system. Because the pseudoparticles form an ideal gas, the quasiparticles are essentially the same as pseudoparticles excited out of the condensate. The holes left behind in the pseudoparticle condensate will be called quasiholes. Hence, the name "pseudo" and "quasi" will be used interchangeably in some cases.

If $\lambda = p/q$, where p and q are relative primes, the motifs for eigenstates can be constructed. The *ones* and *zeros* in the motif mean occupied and unoccupied k_j, respectively. From Eq. (10) the following rules can be deduced for constructing the motif. (i) Total of N *ones* in the motif represent the pseudoparticles. Hence, the charge of pseudoparticle and real particle is the same. (ii) *The allowed number of* zeros *between each pair of* ones *is* $p - 1 + nq$, *where n is an arbitrary non-negative integer.* Of the $p - 1 + nq$ allowed zeros, $p - 1$ consecutive *zeros* are *bound* to each *one* while the rest of them are *unbound*. The consecutive q *unbound zeros* represent a hole. (iii) A cutoff is introduced to make the number of *unbound zeros* finite. The natural unit for the pseudomomenta is then $2\pi/qL$ which is an analog of the flux quantum. To give some examples, the ground state (a), allowed (b), and forbidden (c) states for $\lambda = 3/4$ for $N = 7$ are represented, respectively, by

(a) ...000000001001001001001001001000000000...

(b) ...000000001001001001001001000000010000...

(c) ...000000001001001000001001001001000000... .

The motif (c) violates rule (ii) and hence not allowed. The *bound zeros* which cause spacings between the *ones* can be viewed as flux attached to the pseudoparticles.

The motif thus constructed is a powerful tool in visualizing the fractional statistics. Destroying a particle (or turning a *one* into a *zero*) creates p extra *unbound zeros* and in order to have integer number of holes in the condensate [rule (ii)], a minimum of $q - 1$ extra quasiparticles must be excited out of the condensate leaving behind a total of qp *unbound zeros* which break up into p holes. Hence, q particles leave behind p holes. This is a generalization of the Pauli exclusion principle which explains intuitively why the density-density correlation function has q quasiparticles and p quasiholes when $\lambda = p/q$. It will be shown explicitly that the only states that contribute to the DDDCF are indeed this minimal excitation.

Before I begin to discuss the method used to calculate the DDDCF, some notations need to be defined. First, a diagram $\mathcal{D}(\kappa)$ is defined to be rows and columns of boxes labeled by $\{(i,j) : 1 \le i \le l(\kappa), 1 \le j \le \kappa_i\}$, where $l(\kappa)$ denotes the number of nonzero κ_j. The label i and j are row and column indices of the diagram with $l(\kappa)$ rows of lengths κ_j. Second, the conjugate of κ denoted by κ' is obtained from κ by changing all the rows to columns in nonincreasing order [15]. Each row (column) corresponds to the quasiparticle (hole) excitations. Third, a generalization of factorial is defined by $[a]^\lambda_\kappa = \prod_{(i,j)\in\kappa}[a + (j-1)/\lambda - (i-1)]$. Using the above notations, the density-density correlation function at *finite* N and L is given by

$$\langle 0|\rho(x,t)\rho(0,0)|0\rangle = \frac{1}{L^2}\frac{2}{\lambda^2}\sum_\kappa \frac{|\kappa|^2}{j^\lambda_\kappa}\frac{([0']^\lambda_\kappa)^2[N]^\lambda_\kappa}{[N+1/\lambda-1]^\lambda_\kappa} \times \cos(2\pi|\kappa|x/L)e^{-itE_\kappa}, \tag{11}$$

where $E_\kappa = (2\pi/L)^2\sum_{j=1}^N \kappa_j^2 + \lambda\sum_{j=1}^N(N+1-2j)\kappa_j$ and $j^\lambda_\kappa = \prod_{(i,j)\in\kappa}[\kappa'_j - i + 1 + (\kappa_i - j)/\lambda][\kappa'_j - i + (\kappa_i - j + 1)/\lambda]$. The product in $[0']^\lambda_\kappa$ does not include the pair $(i,j) = (1,1)$. In deriving the finite size correlation function, I use the following two relations [16,17]:

$$\sum_i z_i^n = \frac{n}{\lambda}\sum_{|\kappa|=n}\frac{[0']^\lambda_\kappa}{j^\lambda_\kappa}J^{1/\lambda}_\kappa(\{z_i\}), \tag{12}$$

$$\langle\kappa|\kappa'\rangle_\lambda = j^\lambda_\kappa\frac{[N]^\lambda_\kappa}{[N+1/\lambda-1]^\lambda_\kappa}\delta_{\kappa,\kappa'}. \tag{13}$$

The first relation can be used to expand the reduced density operator $\rho(x) = (1/L)\sum_{j=1}^N[\delta(x-x_j) - 1]$ in terms of the Jack polynomial. The second relation gives the normalization constants of all the excited states parametrized by κ. The DDDCF is now easy to calculate since the eigenstate κ evolves in time only with a phase $\exp(-itE_\kappa)$.

For $\lambda = p/q$, the coefficient $[0']^\lambda_\kappa$ is zero if $\mathcal{D}(\kappa)$ consists of more than p columns or more than q rows. Hence, in the thermodynamic limit a local density operator acting on the ground state provokes only the minimal excitations consisting of q and p quasiparticles and holes.

The large N expansion of Eq. (11) can be carried out at fixed density $\rho = N/L$, and the leading order term corresponding to the thermodynamic limit is given by Eq. (2). In this limit, a new super selection rule emerges and suppresses the states with $|\kappa_i - \kappa_j| \approx O(1)$ to $O(1/N^{2\lambda})$ or $O(1/N^{2/\lambda})$ depending on the value of λ. This means that the states with quasiparticles and quasiholes with same momenta (velocities) are suppressed. There are some exotic exceptions to this rule. The details will be published elsewhere.

The form of the ground state wave function, Eq. (8), has lead Haldane to suggest that while the apparent statistics can be modified with a singular gauge transformation, the "natural" statistics of the CSM are fractional, and that the particle excitations carry charge 1 and flux $\pi\lambda$ and the hole excitations charge $-1/\lambda$ and flux $-\pi$ [18]. Indeed, if a singular gauge transformation [19] is applied, the ground state wave function can be rewritten as

$$\Psi_0 = \prod_{j>l}\frac{(z_j - z_l)^\beta}{|z_j - z_l|^\beta}|z_j - z_l|^\lambda\prod_k z_k^{-\beta(N-1)2}, \tag{14}$$

where the apparent statistical parameter is now $\theta = \pi\beta$. Unlike in the two-dimensional case, the transformed wave function remains as the ground state of the original Hamiltonian, and so are all the other eigenstates. The DDDCF is also unchanged. The hole propagator, however, will be different for different choices of statistics (see, for example, [20]). Hence, in order to calculate the hole propagator, it will be necessary to adopt Haldane's natural exchange statistics for the CSM particles.

In order to consider the fractional exchange statistics, it is convenient to multiply the wave function Ψ by an "ordering function" $\varphi_B(x_{B_1}, \ldots, x_{B_N})$ which is just a bookkeeping device for the phase factor that depends on the braiding B and is set to unity for the fundamental region $x_1 < x_2 < \cdots < x_N$ with no braiding. (In the case of fermions the function is just a product of Grassmann numbers.) Since the ordering function automatically keeps track of all the exchange phases, the particle exchange operator P_{ij} acting on the full wave function amounts to simply exchanging the indices i and j. If the natural statistics is chosen, the phases arising from φ and Ψ can always be set to cancel each other.

In the original CSM there is no physical process which allows particles to exchange (i.e., the wave function vanishes like $|x_i - x_j|^\lambda$as $x_i \rightarrow x_j$). A few years ago, Polychronakos [21] solved this problem by introducing an analog of the Yangian generator [4]

$$\pi_j = p_j + (i\pi\lambda/L)\sum_{i\neq j}\cot[\pi(x_i - x_j)/L]P_{ij}, \qquad (15)$$

where p_j is the ordinary momentum operator, and showed that the Hamiltonian $H = \sum_j \pi_j^2$ is fully integrable and is same as the CSM up to a modification $\lambda(\lambda - P_{ij})$ plus some trivial constant. The new operator π_j is the momentum operator for the pseudoparticles, and the momenta $\sum_j k_j$ corresponds to the eigenvalues of $\sum_j \pi_j$. This new Hamiltonian should be considered as the model of one-dimensional anyons with fractional exchange statistics.

The single particle destruction operation on the ground state of $N + 1$ anyons is, then, given by

$$\Psi(x)|0\rangle_{N+1} = z^{-\lambda N/2}\prod_{j=1}^{N}(z - z_j)^\lambda z_j^{-\lambda/2}|0\rangle_N, \qquad (16)$$

where $z = \exp(i2\pi x/L)$. Now, a similar technique used for the DDDCF can be employed to solve for the hole propagator. In this case, the contributing partitions have no more than p columns and $q - 1$ rows (i.e., p quasiholes and $q - 1$ quasiparticles). Therefore, the natural exchange statistics of the real particles is fully compatible with the exclusion statistics of the elementary excitations.

In the thermodynamic limit the one-particle Green's function (hole propagator) is given by

$$\langle 0|\Psi^\dagger(x,t)\Psi(0,0)|0\rangle = \rho D\prod_{i=1}^{q-1}\left(\int_0^\infty dx_i\right)\prod_{j=1}^{p}\left(\int_0^1 dy_j\right) \times F(q - 1, p, \lambda|\{x_i, y_j\}) \times e^{i\left((Q-Q_0)x-(E-\mu)t\right)}, \qquad (17)$$

where the chemical potential $\mu = (\pi\lambda\rho)^2$ and the back flow $Q_0 = \pi\lambda\rho$. $F(q - 1, p, \lambda|\{x_i, y_j\})$ is still given by Eq. (7) and D by

$$D = \frac{\lambda^{2p(q-1)}\Gamma(p)}{\Gamma(\lambda)(q - 1)!p!}A(q - 1, p, \lambda). \qquad (18)$$

Q and E are same as before except for the number of x_j's. At integer values of λ (i.e., $q = 1$ case where only quasiholes are excited), based on the equal-time results of Forrester [22] Haldane made a conjecture [18] which agrees with this formula. I conjecture that the minimal form factor for any two-point correlation function is given by $F(m, n, \lambda|\{x_i, y_j\})$ if the intermediate states involve only m quasiparticles and n quasiholes.

In conclusion, the CSM is shown to possess the fractional *exclusion* and *exchange* statistics. The motifs representing the full spectrum are constructed and used to demonstrate the exclusion statistics, explicitly. The fractional statistics in the CSM is also confirmed by calculating the exact dynamical density-density correlation function and the one-particle Green's function (hole propagator) at any rational interaction coupling constant using the theory of Jack symmetric orthogonal polynomials. The details of the calculation will be published elsewhere.

While this Letter focuses on the fractional statistics aspect of the CSM, the method for calculating the correlation functions developed here could be of interest to a wide variety of people working on the disordered electronic system, the quantum chaos, the random matrix theory, 2D QCD, etc.

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