Generalized Ensemble of Random Matrices

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A random matrix ensemble incorporating both Gaussian unitary ensemble and Poisson level statistics while respecting $U(N)$ invariance is proposed and shown to be equivalent to a system of noninteracting, confined, one-dimensional fermions at finite temperature.

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Wigner's [1] suggestion to model the level statistics of complex quantum systems by random matrix models of very large dimensions is well known. In applications as diverse as two-dimensional gravity [2], topological field theory [3], quantum chaos [4], and mesoscopic physics [5] different kinds of universal features emerge, all represented by particular types of random matrix ensembles. The analysis of these models seems to lead one to classical [2,3] and quantum integrable systems [6] of very special mathematical structure, and many unexpected relations make their appearance [7].

Random matrix models exhibit a large degree of robustness [8] and the search for different kinds of universality has produced a wealth of results in two-dimensional quantum gravity [2] and in topological field theory [3]. In quantum chaos and mesoscopic physics there are physical reasons to expect more universality classes than presently known [9–12]. This Letter takes a small step in this direction.

For reasons of concreteness and mathematical simplicity, we restrict our attention to a disordered system without time-reversal symmetry (e.g., in the presence of a magnetic field) and ask whether random matrix models can account for known departures from GUE (Gaussian unitary ensemble) level statistics [10–13]. On the one hand, the class of matrix models with GUE level statistics has been shown recently to be very wide [8,14] and also to agree well with the properties in the energy region of extended states [15], but, on the other hand, disordered systems have energy regions where the states are localized and the level statistics is Poissonian. To be sure, even localized states, once sufficiently close energetically, eventually repel each other because the exponentially small overlap can be outweighed by exceedingly small energy denominators. It is the typical scale of energy differences needed for level repulsion to be felt that changes dramatically in transition from extended to localized regimes.

Recently, an attempt to incorporate both statistics in a random matrix model was made [12]: The basic idea was that in the localized regime the Hamiltonian prefers the site basis, and hence the random matrix model was made to break the $U(N)$ invariance which represents the lack of basis preference in the GUE ensemble. For sufficiently strong breaking, approximate Poisson statistics could be obtained. The explicit breaking of $U(N)$ invariance makes the mathematical treatment and the study of robustness difficult. While the models we are discussing cannot be fully accurate representations of the mesoscopic physics since we will not introduce any explicit dependence on dimension, we feel that exploring the possibility for constructing such an enlarged universality class is of more general interest and holds the potential for wider applicability than just to mesoscopic physics.

Our main new observation is that basis preference, while needed, does not necessarily force the abolishment of $U(N)$ symmetry. In a way vaguely reminiscent of spontaneous symmetry breaking, we argue that what is important is the preference of *some* basis but it is immaterial which basis this is. Averaging over all preferred bases we recover $U(N)$ invariance and still manage to evade GUE statistics if a certain parameter is made strong enough.

Imagine that the preferred basis is represented by an $N \times N$ unitary matrix V. This representation has N redundant phases, and it is therefore convenient to consider instead the unitary matrix $U = VDV^\dagger$ where $D_{ij} = \delta_{ij} e^{i\theta_i}$ ($|\theta_i| \leq \pi$, $i = 1, 2, \ldots, N$). For U to define uniquely a preferred basis we need the $e^{i\theta_j}$'s to be all different. In order to ensure this upon averaging we would like the weight to contain sufficient repulsion between the $e^{i\theta_j}$'s. Let the Hamiltonian be represented by a Hermitian matrix H. For fixed U we consider the unnormalized distribution of H given by

$$\mathcal{P}_U(H) d^{N^2}H \propto e^{-\mathrm{Tr}H^2} e^{-b\mathrm{Tr}([U,H][U,H]^\dagger)} d^{N^2}H . \quad (1)$$

Here, $d^{N^2}H = \prod_{i=1}^{N} dH_{ii} \prod_{i>j} d\mathrm{Re}(H_{ij}) d\mathrm{Im}(H_{ij})$ and $b > 0$. The b-dependent term tries to align H with U and hence prefers the basis V. We now average over U with the invariant $U(N)$ Haar measure dU. This induces enough repulsion between the eigenvalues of U [1,16] to ensure that a unique preferred basis almost always exists. Our ensemble therefore is

$$\mathcal{P}(H) d^{N^2}H = C'(N,b) e^{-(2b+1)\mathrm{Tr}H^2} \times \left[\int dU\, e^{2b\mathrm{Tr}(UHU^\dagger H)} \right] d^{N^2}H , \quad (2)$$

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where $C'(N,b)$ is a normalization constant. The integral over U can be done using the famous formula [17]

$$\int dU \exp[\mathrm{Tr}UAU'B] \propto \frac{\mathrm{Det}_{ij}[\exp(a_i b_j)]}{\prod_{i>j}(a_i - a_j)\prod_{i>j}(b_i - b_j)},$$

where the a_i and b_j are the eigenvalues of the Hermitian matrices A and B. Using this integration formula with $A = B/2b = H$ we obtain, after absorbing the b-dependent part of the prefactor in Eq. (2) into the determinant,

$$\mathcal{P}(H)d^{N^2}H \propto \frac{\mathrm{Det}_{ij}\left(e^{-\left(b+\frac{1}{2}\right)\left(x_i^2+x_j^2\right)+2bx_ix_j}\right)}{\prod_{i>j}(x_i - x_j)^2} d^{N^2}H.$$

The measure $d^{N^2}H$ can be written as $\prod_{i>j}(x_i - x_j)^2 \times \prod_i dx_i\, d\mu$ where $d\mu$ is the angular part which can be simply integrated out. We then obtain the joint probability density function for the eigenvalues

$$P(x_1, x_2, \ldots, x_N)\prod_i dx_i = C(N,b)\,\mathrm{Det}_{ij}\left(e^{-\left(b+\frac{1}{2}\right)\left(x_i^2+x_j^2\right)+2bx_ix_j}\right)\prod_i dx_i. \tag{3}$$

$P(x_1, x_2, \ldots, x_N)$ in Eq. (3) coincides [18–20] with the diagonal element in the "coordinate representation (x)" of the density matrix of N noninteracting fermions of mass m, in one dimension, moving in a confining harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ at finite temperature,

$$\begin{aligned} P(x_1, x_2, \ldots, x_N) &= \mathcal{N}\mathrm{Det}_{ij}\left(\langle x_i|\exp\left\{-\beta\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}\right)\right\}|x_j\rangle\right) \\ &= \mathcal{N}\mathrm{Det}_{ij}\left(\exp\left\{-\frac{m\omega}{2\hbar}\coth(\hbar\omega\beta)\,(x_i^2 + x_j^2) + \frac{m\omega}{\hbar}\frac{x_ix_j}{\sinh(\hbar\omega\beta)}\right\}\right), \end{aligned} \tag{4}$$

where $\mathcal{N}$ is a normalization factor, $\beta\hbar\omega = \mathrm{arcosh}(1 + \frac{1}{2b})$, and the length unit is fixed by $m\omega/\hbar = \sqrt{1 + 4b}$. This means that the mass and temperature are related by $m\omega/\hbar = \coth(\hbar\omega\beta/2)$. The GUE ensemble is obtained in the limit $\hbar\omega\beta \to \infty$, $m\omega/\hbar \to 1$ [implied by $b \to 0$, GUE statistics in Eq. (3)], and complete disappearance of level repulsion is obtained in the limit $\hbar\omega\beta \to 0$, $m\omega/\hbar \to \infty$ [$b \to \infty$, Poisson statistics in Eq. (3)].

When N is large, it is much more convenient to analyze the grand canonical version of the ensemble (3). Indexing the one-particle states by α, denoting their energies by ϵ_α, and the corresponding wave functions by $\psi_\alpha(x)$, one immediately can write down the expression for the local density $\bar{n}(x)$:

$$\bar{n}(x) = \sum_\alpha \frac{|\psi_\alpha(x)|^2}{e^{\beta(\epsilon_\alpha - \mu)} + 1}. \tag{5}$$

The chemical potential μ is fixed by N via $N = \sum_\alpha [e^{\beta(\epsilon_\alpha - \mu)} + 1]^{-1}$.

For large N one can use semiclassical forms for $\psi_\alpha(x)$ as long as x is away from the edges of the support of $\bar{n}(x)$ and replace the sum over α by an integral. A simple computation gives then

$$\bar{n}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \frac{1}{1 + [e^{\beta\hbar\omega N} - 1]^{-1} e^{\beta\left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2\right)}}. \tag{6}$$

Equation (6) already exhibits both limiting situations of GUE and of Poisson statistics: For b growing with N slower than N^2, $\beta N \to \infty$, the gas becomes completely degenerate, and one obtains the GUE semicircle law

$$\bar{n}(x) = \frac{\sqrt{4b+1}}{\pi}\left[\frac{2N}{\sqrt{4b+1}} - x^2\right]^{\frac{1}{2}}. \tag{7}$$

For b growing with N faster than N^2, $\beta N \to 0$, a Gaussian distribution is obtained:

$$\bar{n}(x) = \frac{N}{\sqrt{\pi}} e^{-x^2}. \tag{8}$$

Correlations can also be easily calculated. Consider the 2-point connected correlation:

$$\overline{\delta n(x)\delta(n)(x')} = -\left|\sum_\alpha \bar{n}_\alpha \psi_\alpha^*(x)\psi_\alpha(x')\right|^2, \qquad x \neq x', \tag{9}$$

with $\bar{n}_\alpha = [e^{\beta(\epsilon_\alpha - \mu)} + 1]^{-1}$. Writing $x = \bar{x} + \frac{1}{2}\Delta x$, $x' = \bar{x} - \frac{1}{2}\Delta x$ with $0 < |\Delta x| \ll \bar{n}(d\bar{n}/dx)^{-1}$, we derive for $\beta N \to \infty$:

$$\frac{\overline{\delta n(x)\delta n(x')}}{[\bar{n}(\bar{x})]^2} = -\left[\frac{\sin[\pi\bar{n}(\bar{x})\Delta x]}{\pi\bar{n}(\bar{x})\Delta x}\right]^2 \tag{10}$$

and for $\beta N \to 0$:

$$\frac{\overline{\delta n(x)\delta n(x')}}{[\bar{n}(\bar{x})]^2} = -e^{-2b(\Delta x)^2} \to 0. \tag{11}$$

The last two equations show that the model incorporates GUE and Poisson statistics.

Consider now the intermediate case when, in the $N \to \infty$ limit, $\beta N \to$ const $\equiv A$. If the constant A is small, the average density of levels $\bar{n}(x)$ and the correlations are approximately the same as for $\beta N \to 0$. The case $A \gg 1$ is more interesting: $\bar{n}(x)$ is a complicated function which, for small x, is proportional to $\sqrt{A - x^2}$ and, for large x, is proportional to $\exp(-x^2)$ ["small" and "large" here mean, respectively, $A - x^2 \gg 1$ and $-(A - x^2) \gg 1$]. The correlation also changes when one moves from the center of the band towards the edges: At the edges (i.e., $x \gg A$) the

levels are essentially uncorrelated, whereas at the band center ($x \approx 0$) the correlation function is given by

$$\frac{\overline{\delta n(x)\delta n(x')}}{[\overline{n}(x)]^2} = -\left(\frac{\sin \pi u}{\pi u}\right)^2 \left[\frac{\pi^2 u/2A}{\sinh(\pi^2 u/2A)}\right]^2, \quad (12)$$

where $u \equiv \overline{n}(x)\Delta x$ is the energy separation measured in units of the averaged level spacings. While a sequence of roughly A levels obeys GUE statistics, the correlations between more distant levels rapidly diminish. This is in sharp contrast with the case $\beta N \to \infty$ ($\beta N \to 0$) where, in the limit $N \to \infty$, an arbitrary long sequence of levels obeys the GUE (Poisson) statistics. In this sense the parameter βN of our model can be identified with the dimensionless conductance g. $g \to \infty$ corresponds to metallic behavior, $g \to 0$ corresponds to an insulator, and $g \to$ const is identified with a mobility edge situation, where a third universal statistics, intermediate between the GUE and Poisson, is observed [10–12]. In that latter case $\overline{\Delta N_\Gamma^2} \approx \overline{N}_\Gamma/2A$ where $\overline{\Delta N_\Gamma^2}$ describes fluctuations within an energy interval Γ with an average number of levels equal to $\overline{N}_\Gamma$ ($\overline{N}_\Gamma \gg A$ is assumed). The linear relation between $\overline{\Delta N_\Gamma^2}$ and $\overline{N}_\Gamma$ is a consequence of the exponential decay of correlations in Eq. (12). In Ref. [11] it was argued that a linear dependence is ruled out in the universal regime by a sum rule expressing the conservation of the total number of levels N. In our analysis of the matrix model we found that in the large N limit it is correct to pass from the canonical description, fixed number of levels or fermions, to the grand canonical one, arbitrary number of levels in the presence of a "chemical potential." Therefore, the overall level number conservation cannot affect universal features of the statistics of finite sequences of levels in the thermodynamic limit.

A guess for the enlarged universality class of these models is the following: add more $U(N)$ invariant function of U and H to the exponent in Eq. (1) and average over U. While strict $U(N)$ invariance is useful, models with explicit breaking [12] are not excluded and can easily turn out to be controlled by $U(N)$-symmetric fixed points. In this context it is amusing to note that if, in a reversal of roles, one integrates over H in (1) and views the resulting distribution of U as a generalized circular Dyson-type ensemble [16], one recovers a "circular" version of the eigenvalue distribution arrived at in Ref. [12]:

$$P_\theta(\theta_1, \ldots, \theta_N) \prod_{i=1}^{N} d\theta_i = C''(N,b) \times \prod_{i>j} \left[\frac{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)}{1 + 4b \sin^2\left(\frac{\theta_i - \theta_j}{2}\right)}\right] \prod_{i=1}^{N} d\theta_i . \quad (13)$$

For numerical purposes one may wish to stick to the canonical ensemble; expressions for $\overline{n}(x)$ and for $\overline{\delta n(x)\delta n(x')}/[\overline{n}(x)]^2$ can be obtained in terms of infinite sums that converge fast as long as βN does not grow to infinity [20,21]. For example, the exact formula for $\overline{n}(x)$, in Eq. (5), is

$$\overline{n}(x) = \sum_{\alpha=1}^{\infty} |\psi_\alpha(x)|^2 \left\{ \sum_{r=0}^{\infty} (-1)^r q^{(N-\alpha)r + \frac{1}{2}r(r+1)} \times \prod_{\ell=N+1}^{N+r} (1 - q^\ell)^{-1} \right\}, \quad (14)$$

where $\epsilon_\alpha = \hbar\omega(\alpha - \frac{1}{2})$, $q = e^{-\beta\hbar\omega} = [1 + 1/2b + \sqrt{1/b + 1/4b^2}\,]^{-1}$, and $\prod_{\ell=N+1}^{N}(1 - q^\ell)^{-1} \equiv 1$.

Equation (2) can be viewed as a degenerate case, in one space-time dimension, of "induced QCD" models [22]. Generalizations of the ensemble discussed here to the orthogonal or symplectic case are likely to produce technical complications similar to the ones encountered in the study of random unoriented two-dimensional manifolds with or without matter [23].

Let us mention the widely studied ensemble of banded matrices (Ref. [13] and references therein). For such matrices, in contrast to the models considered here, only elements within a "band" of width W near the diagonal are appreciably different from zero. Statistical properties of the ensemble depend on the value of the parameter N/W^2, which plays a role similar to our parameter b. When N increases, for fixed W, the ensemble describes a quasi-one-dimensional disordered system with localized eigenfunctions [13]. It should also be possible, by an appropriate scaling of W with N, to obtain a limiting eigenvalue distribution which is neither Wigner-Dyson nor Poisson. To our knowledge, there are no detailed analytical results for the eigenvalue distribution comparable in their generality to our Eq. (3), which gives the distribution for any value of N and b.

In summary, the random matrix model proposed in Eqs. (2) and (3) is an extension of the standard Gaussian unitary ensemble with a level distribution that interpolates between Wigner-Dyson and Poisson statistics. $U(N)$ invariance is preserved, and the model is equivalent to a system of one-dimensional fermions in a harmonic potential at finite temperature. Possible applications are to modeling of level statistics in quantum chaos, mesoscopic physics, and disordered electronic systems. For example, in the latter case, the proposed ensemble can be driven from metallic to insulator behavior by varying the parameter b.

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[1] E. P. Wigner, Proc. Cambridge Philos. Soc. **47**, 790 (1951); M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991), 2nd ed.

[2] E. Brézin and V. Kazakov, Phys. Lett. B **236**, 2125 (1990); M. R. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990); D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 27 (1990).

[3] E. Witten, Nucl. Phys. **B340**, 281 (1990).

[4] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer Verlag, New York, 1990); T. M. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1984); Phys. Lett. **108A**, 183 (1985).

[5] L. P. Gorkov and G. M. Eliashberg, Sov. Phys. JETP **21**, 940 (1965); Y. Imry, Europhys. Lett. **1**, 249 (1986); B. L. Altshuler and B. I. Shklovskii, Sov. Phys. JETP **64**, 127 (1986).

[6] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993).

[7] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **72**, 64 (1994).

[8] E. Brézin and A. Zee, Nucl. Phys. **B402 [FS]**, 613 (1993).

[9] T. M. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1984); Phys. Lett. **108A**, 183 (1985).

[10] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11 487 (1993).

[11] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler, and A. G. Aronov, Phys. Rev. Lett. **72**, 888 (1994).

[12] J.-L. Pichard and B. Shapiro, J. Phys. **4**, 623 (1994).

[13] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **69**, 1093 (1992); Phys. Rev. Lett. **71**, 412 (1993).

[14] E. Brézin and A. Zee, Ecole Normale Supérieure report, 1993 (to be published).

[15] B. D. Simons and B. L. Altshuler, Phys. Rev. Lett. **70**, 4063 (1993); A. Szafer and B. L. Altshuler, Phys. Rev. Lett. **70**, 587 (1993).

[16] F. J. Dyson, J. Math. Phys. **3**, 1199 (1962).

[17] C. Itzykson and J. B. Zuber, J. Math. Phys. **21**, 411 (1980).

[18] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, Commun. Math. Phys. **59**, 35 (1978).

[19] G. Parisi, Phys. Lett. B **238**, 213 (1990).

[20] D. Boulatov and V. Kazakov, Nucl. Phys. (Proc. Suppl.) **25A**, 38 (1992).

[21] M. Caselle, A. D'Adda, and S. Panzeri, Phys. Lett. B **293**, 161 (1992).

[22] V. A. Kazakov and A. A. Migdal, Nucl. Phys. **B397**, 214 (1993).

[23] E. Brézin and H. Neuberger, Nucl. Phys. **B350**, 513 (1990); H. Neuberger, Phys. Lett. B **257**, 45 (1991).