Circuit Theory of Andreev Conductance

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Conductance of small normal metal structures adjacent to a superconductor is determined by coherent Andreev reflection. We show that under certain limitations the conductance can be found by means of an extended circuit theory. The theory deals with two types of elements: tunnel junctions and diffusive conductors and provides the basis for practical calculations. A new device proposed illustrates the advantages of the theory.

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What is the resistance of a normal metal structure in series with a superconductor? Common sense readily prompts that it must be the resistance of the normal structure. It is known from early days of superconductivity that this prompt is misleading. The reason for this is the energy gap in the superconductor. Normal electrons having energies below the gap thus cannot drain off into superconducting part of the system. This implies that the only mechanism of low voltage, low temperature charge transport is Andreev reflection [1]: normal electron reflects from the normal-superconducting (NS) interface as a hole transferring a charge $2e$ to the superconductor. We will refer to the conductivity of the NS system as *Andreev* conductivity, to stress the change brought by the superconductor.

Early studies mostly concentrated on the resistance of a NS interface itself [2]. Lately it has been understood that the Andreev reflection is an essentially coherent phenomenon (see, e.g., [3,4]) and interference between the incoming electron and outgoing hole persists in the normal metal at mesoscopically large distances from the interface. Therefore Andreev conductance of a sufficiently small structure is a property of the whole structure but not only the NS interface. The space scale involved is in fact the same characterizing the proximity effect. If we assume diffusive electron transport, this scale $\xi \simeq \sqrt{\mathcal{D}\hbar/\varepsilon}$, $\mathcal{D}$ being diffusivity, ε being the typical energy of the electron (or the hole). A good estimation for ε is $\max\{T, eV\}$, V being the voltage applied to the structure [5]. The recent advantages in microtechnology make possible the fabrication of controlled devices of such a size. The fresh example is an experimental realization of a NS quantum interference device [6] described in practical terms in Ref. [4]. There is an urgent need in theoretical developments providing for quantitative analysis and design of such devices.

The problem has been tackled recently by different methods. The authors of Ref. [7] use the traditional nonequilibrium superconductivity approach [8]. Many interesting results can be obtained in the tunneling Hamiltonian framework [4]. The problem can be treated within a scattering formalism [9]. Unfortunately direct computer simulations [10] can only deal with unrealistically small structures. Alternative analytical work stopped due to insufficient knowledge about transmission matrix of a mesoscopic conductor. Despite recent progress in this direction [11], the problems arising are much more difficult to handle than the original one.

I am certainly in favor of the first approach mentioned since it is the most general one and picks up only relevant physics. We show that under quite general assumptions the technical complications may be lifted and the problem can be formulated in terms of a comprehensive circuit theory. From now on, the calculation of Andreev conductance is a subject of technical rather than theoretical physics.

The paper is organized as follows. First we show how to get to circuit theory starting from equations of Refs. [7,8]. The necessary limitations will be spelled out. Then the rules of the theory will be formulated. Taking simple examples, we demonstrate consistency with other approaches. To show the power of the method developed, a new device will be considered in the final part of the paper.

Let us consider a normal metal structure adjacent to one or more superconducting terminals. There may be several normal terminals biased at different voltages. There may be interfaces inside the structure such as tunnel junctions. Physics of electric conduction in such a system can be treated with nonequilibrium superconductivity equations. Those are written for Keldysh Green's functions in coinciding points [7,8]. Those are 4×4 matrices depending in a stationary case on space coordinates and energy. They can be subdivided onto 2×2 matrices made up from usual and anomalous Green functions (we use "check" for 4×4 and "hat" for 2×2 matrices):

$$\check{G} = \begin{pmatrix} \hat{G}^A & \hat{G} \\ 0 & \hat{G}^R \end{pmatrix} \; \hat{G}^A = \begin{pmatrix} G^A & F^A \\ -F^{+A} & G^{+A} \end{pmatrix}. \quad (1)$$

Here advanced and retarded functions determine the characteristics of the energy spectrum of the system in a given point whereas $\hat{G}$ sets particle distribution over these energy states and thus directly related to the electric current and other physical quantities. It is assumed that the size

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of the system at least in a transport direction greatly exceeds Fermi wavelength and elastic mean free path, this makes diffusive approximation sensible. In this approximation, nonequilibrium superconductivity equations resemble a standard diffusion equation and read [7,8]

$$\frac{\partial}{\partial x^\alpha}\mathcal{D}(x)\check{G}\frac{\partial}{\partial x^\alpha}\check{G} + [\check{H},\check{G}] = 0, \tag{2}$$

$$\check{H} = \epsilon\check{\tau}_z + i\check{\tau}_x \mathrm{Re}[\Delta(x)] + i\check{\tau}_y \mathrm{Im}[\Delta(x)].$$

Here Δ is superconducting pair potential, τ are Pauli matrices, and square brackets denote a commutator of two matrices. A unitary condition holds for $\check{G}$: $\check{G}^2 = \check{1}$. The similarity becomes closer if it is possible to skip the rightmost term in (2). We do so, and now we spell limitations under which it is a true thing to do. The left term is of the order of $\mathcal{D}/L^2$, L being the system size, or, more precisely, the size at which the resistance of the structure is being formed. In the normal metal $\Delta \equiv 0$ and the right term is of the order of ε. Thus we need a sufficiently small system: $L \ll \xi$. It implies the following: (a) The temperature is low enough: $T \ll \mathcal{D}/L^2, T \ll \Delta$. (b) The voltage is low enough: $V \ll \mathcal{D}/L^2, V \ll \Delta$. The last limitation below is given by the fact that we use stationary equation. If there had been several superconducting terminals in the structure biased by different voltages, it would have given rise to nonstationary Josephson-like effect which we do not intend to treat here. So that, (c) all superconducting terminals are at the same voltage. Let us set this voltage to zero.

Under these limitations, (2) takes the compact form of a conservation law for a matrix current:

$$\frac{\partial}{\partial x^\alpha}\check{j}^\alpha(x) = 0, \quad \check{j}^\alpha(x) = \sigma(x)\check{G}\frac{\partial}{\partial x^\alpha}\check{G}. \tag{3}$$

Here $\sigma(x)$ is specific conductivity in the normal state. We made use of the fact that it is directly proportional to diffusivity. If there are interfaces in the structure, they give boundary conditions for (3). These conditions have been derived in [12] and can be written in a comprehensive form

$$N^\alpha \check{j}^\alpha(x) = g(x)[\check{G}_1(x), \check{G}_2(x)], \tag{4}$$

N^α being a vector normal to the interface at a point x, $g(x)$ being the conductance of the interface per unit area at the same point, and $1,2$ refer to different sides of the interface.

To proceed, we recall the equations describing the conductivity of the metal structure in normal state. The following are evident:

$$\frac{\partial}{\partial x^\alpha}j^\alpha(x) = 0, \quad j^\alpha(x) = -\sigma(x)\frac{\partial}{\partial x^\alpha}u, \tag{5}$$

j being electric current density, u being electrostatic potential at a point x. The current through the interface is proportional to the voltage drop:

$$N^\alpha j^\alpha(x) = g(x)[u_1(x) - u_2(x)]. \tag{6}$$

One knows how to get from Eqs. (5) and (6) to circuit theory. One separates the structure on the resistive elements finding from Eqs. (5) and (6) how the current through each element relates to the voltage difference across it. Conservation of the current in nodal points of resulting network (second Kirchhoff rule) will give the relations necessary to find the voltage and the current in every point of the network.

The key idea of the innovation being currently presented is to do the same with Eqs. (3) and (4). The only distinction is that we have to deal with matrix currents and matrix drops.

We can simplify Eqs. (3) and (4) further. The goal will be to separate equations determining the equilibrium spectral properties of the structure from those determining the propagation of nonequilibrium carriers in the structure. To do so, we introduce the standard [8] parametrization of the matrix $\hat{G}$:

$$\hat{G} = \frac{1}{2}\Big\{f(\epsilon)\Big[\hat{G}^R(1+\hat{\tau}_z) - (1+\hat{\tau}_z)\hat{G}^A\Big] + f(-\epsilon)\Big[\hat{G}^R(-1+\hat{\tau}_z) - (-1+\hat{\tau}_z)\hat{G}^A\Big]\Big\}, \tag{7}$$

$f(\epsilon)$ can be associated with the distribution function of the quasiparticles. We substitute this into Eqs. (3) and (4) getting equations for $f(\epsilon)$. In the normal terminals, $f(\epsilon) = \tanh[(\epsilon - eV_i)/2T]$, V_i being the voltage of the ith terminal. Since Eq. (3) does not depend explicitly on energy, the same is true for equations for $f(\epsilon)$. We can thus integrate them over energy expressing current in terms of $\zeta(x) = \int d\epsilon[f(\epsilon) - \tanh(\epsilon/2T)]/e$, $\zeta(x)$ measuring a deviation of the quasiparticle distribution from the equilibrium.

We note that at zero energy $\hat{G}^A$, this greatly simplifies the resulting equations. It is convenient to parametrize $\hat{G}^R = \mathbf{s}\hat{\tau}, \mathbf{s}^2 = 1$ (we will use boldface for vectors in the space of Pauli matrices keeping Greek indexes for usual ones). We will call $\mathbf{s}$ a spectral vector since $\hat{G}^R(x)$ determines local spectral properties of the metal. The spectral vector has a simple physical meaning: its z component determines a factor by which the local density of states at zero energy is reduced in comparison with that one in the normal metal. Its latitude shows what is the phase of the Cooper pair amplitude induced in a given point x.

Taking all this into account, we rewrite general equations (3) and (4) in terms of $\mathbf{s}, \zeta$ [13]:

$$\frac{\partial}{\partial x^\alpha}j^\alpha(x) = 0, \quad j^\alpha(x) = -\sigma(x)\frac{\partial}{\partial x^\alpha}\zeta(x), \tag{8}$$

$$N^\alpha \check{j}^\alpha(x) = g(x)\mathbf{s}_1\mathbf{s}_2[\zeta_1(x) - \zeta_2(x)], \tag{9}$$

$$\frac{\partial}{\partial x^\alpha}\mathbf{j}^\alpha(x) = 0, \quad \mathbf{j}^\alpha(x) = \sigma(x)\mathbf{s}(x)\times\frac{\partial}{\partial x^\alpha}\mathbf{s}(x), \tag{10}$$

$$N^\alpha \mathbf{j}^\alpha = g(x)\mathbf{s}_1\times\mathbf{s}_2. \tag{11}$$

Boundary conditions at the terminals read

$$\mathbf{s} \parallel z, \quad \zeta = V_i \tag{12}$$

at all normal terminal;

$$s_x = \cos\phi, \quad s_y = \sin\phi, \quad \zeta = 0 \tag{13}$$

at all superconducting terminals.

Now it becomes clear what is the structure of the extended circuit theory. Real electric current is associated with ζ; its magnitude is proportional to a drop of ζ over an element. The coefficient of proportionality does not depend on spectral vectors for diffusive conductor and does for tunnel junction. In contrast to a normal circuit theory, we thus have two different kinds of resistive elements. We can say that the induced superconductivity does not change the conductivity of the diffusive parts but renormalizes the tunnel junction conductivities. The spectral vectors determining such a renormalization shall be found from Eqs. (10) and (11). It is important that the spectral vectors can be determined from the circuit theory as well. Solving Eqs. (10) and (11) we find how the spectral current relates to the drop of the spectral vectors over an element. For a diffusive conductor we obtain

$$R_D \mathbf{I} = \frac{\mathbf{s}_1 \times \mathbf{s}_2}{\sqrt{1 - (\mathbf{s}_1 \mathbf{s}_2)^2}} \arccos(\mathbf{s}_1 \mathbf{s}_2). \tag{14}$$

The magnitude of the current is thus proportional to the angle between $\mathbf{s}_1$ and $\mathbf{s}_2$. Similar but different relation holds for a tunnel junction:

$$R_T \mathbf{I} = \mathbf{s}_1 \times \mathbf{s}_2. \tag{15}$$

As in a standard circuit theory, the condition of current conservation in nodal points and terminal conditions complete the set of relations which we need to find spectral vectors.

For practical calculations, it is very instructive to picture a network on a hemisphere, as done in Fig. 2. The coordinate of a point of the network corresponds to the spectral vector in this point.

Summarizing, we give the rules of the resulting circuit theory.

(I) Andreev conductance is the same as for normal circuit with renormalized tunnel conductivities. Renormalization factor is given by scalar product of spectral vectors belonging to two shores of the tunnel junction.

The following rules set spectral vectors:

(II) Spectral vector in a normal reservoir $\parallel z$ (north pole of the hemisphere). Spectral vector in a superconductor $\perp z$ (equator of the hemisphere). Its longitude corresponds to the superconducting phase.

(III) The spectral current is perpendicular to both spectral vectors at the ends of an element. Its magnitude is given by either $I = G_D \phi$ for a diffusive conductor or $I = G_T \sin\phi$ for a tunnel junction, ϕ being the angle between the spectral vectors at the ends.

(IV) *Vector* Kirchhoff rule holds in nodal points of the network. That is, vector spectral current is conserved in the nodal points.

Let us start with the simplest examples (see Fig. 1). Suppose we have a tunnel junction in series with a diffusive conductor. According to (I), the Andreev resistance of the circuit is given by $R_A = R_D + R_T/\cos\alpha$, α being latitude of a spectral vector in the point A. To determine α, one equates the spectral current in the tunnel junction with that one in the diffusive conductor:

$$I = \sin\alpha/R_T = (\pi/2 - \alpha)/R_D. \tag{16}$$

These two relations implicitly determine R_A. The result coincides with that of [11] and, in relevant limits, with results obtained in [4,7].

Another important example is two tunnel junctions in series. We will assume that the metal between the junctions is disordered enough to assure diffusive transport but the resistance of this metal is negligible compared with junction resistances. Again, from the first rule we obtain $R_A = R_1/\cos\alpha + R_2/\sin\alpha$. An equation for α reads $I = \sin\alpha/R_1 = \cos\alpha/R_2$. This yields

$$R_A = \frac{(R_1^2 + R_2^2)^{3/2}}{R_1 R_2}. \tag{17}$$

Let us apply the theory to analysis of a more complex network. Let the current flow from the normal electrode to two superconductors that have different phase (Fig. 2). A dramatic feature of Andreev conductance is its dependence on this phase difference ϕ [4,6,14]. This suggests that the electron coming from the normal side is not reflected as a hole from one particular superconductor but rather from *both*. A novel feature of the device currently proposed is that the tunnel junctions are in all three branches of the circuit. This is practical since in this case the Andreev conductance is of the order of the normal conductance of the circuit provided all three junction resistances are comparable.

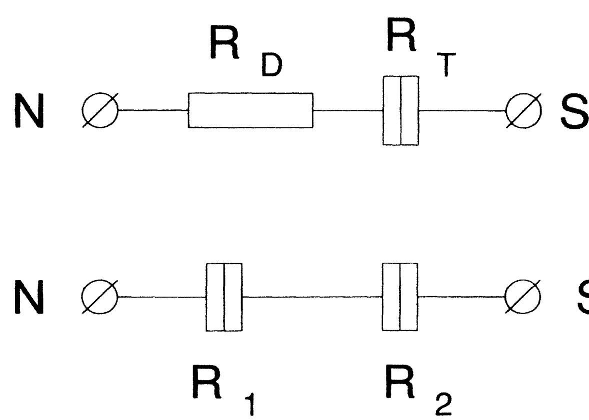

FIG. 1. Simplest NS circuits.

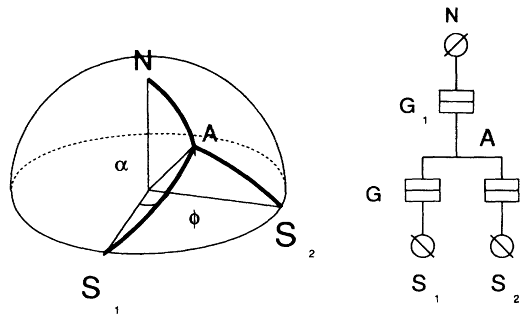

FIG. 2. Three-junction NS quantum interference device. Spectral vectors in the network are mapped onto hemisphere.

Let us map the network onto hemisphere (see Fig. 2). According to (II), spectral vectors at points N, S, S' are set by terminal conditions. The conservation of the spectral current in the nodal point [rule (IV)] determines the spectral vector $\mathbf{s}_A$ in this point:

$$0 = \sum \mathbf{I} = \mathbf{s}_A \times [G_1 \mathbf{s}_N + G(\mathbf{s}_S + \mathbf{s}_{S'})] \tag{18}$$

this yields ($\mathbf{s}^2 = 1$!)

$$\mathbf{s}_A = [G_1 \mathbf{s}_N + G(\mathbf{s}_S + \mathbf{s}_{S'})]/\sqrt{G_1^2 + 2G^2(1 + \cos\phi)}. \tag{19}$$

From this one readily obtains renormalization factors $\mathbf{s}_A \mathbf{s}_{N,S,S'}$ for each junction and calculates the actual conductance

$$G_A = \frac{4G_1^2 G^2 \cos^2(\phi/2)}{\left[G_1^2 + 4G^2 \cos^2(\phi/2)\right]^{3/2}}. \tag{20}$$

Limiting cases are constructive to look at

$$G_A = \frac{4G^2}{G_1} \cos^2(\phi/2) \quad \text{at } G_1 \gg G \tag{21}$$

$$G_A = \frac{G_1^2}{2G\,|\cos(\phi/2)|} \quad \text{at } G_1 \ll G. \tag{22}$$

In both cases, the Andreev conductance is smaller than the normal conductance but the phase dependence is quite different (see Fig. 3). For the first limit, the conductance is maximal at $\phi = 0$ and smoothly goes to zero with increasing ϕ. For the second case, it grows with increasing ϕ reaching the maximum at ϕ very close to π and then sharply drops to zero. This feature of the device proposed makes it very practical for measuring small phase differences.

In conclusion, the circuit theory has been constructed for Andreev conductance of arbitrary complex normal-superconducting networks. It allows us to clarify the complex theoretical constructions used previously and to design new interference devices.

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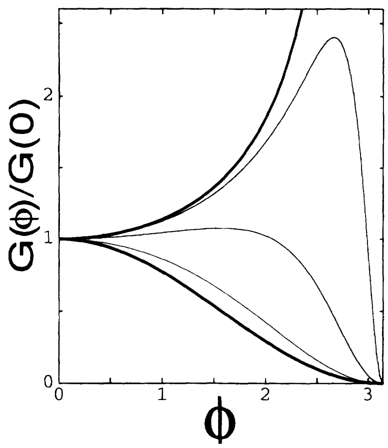

FIG. 3. Phase dependence of three-junction NS quantum interference device conductivity. The parameter G/G_1 increases from the lowermost to the uppermost curve taking values 0.01, 0.333, 1, 3, and 100.

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