Mesoscopic Transport through Chaotic Cavities: A Random S-Matrix Theory Approach

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(Received 15 March 1994)

We deduce the effects of quantum interference on the conductance of chaotic cavities by using a statistical ansatz for the S matrix. Assuming that the circular ensembles describe the S matrix, we find that the conductance fluctuation and weak-localization magnitudes are universal: they are independent of the size and shape of the cavity if the number of incoming modes, N, is large. The limit of small N is more relevant experimentally; here we calculate the full distribution of the conductance and find striking differences as N changes or a magnetic field is applied.

PACS numbers: 72.20.My, 05.45.+b, 72.15.Gd

The effect of quantum interference on transport through microstructures has been intensively investigated and is one of the main subjects of mesoscopic physics [1]. For diffusive transport in disordered structures, both microscopic perturbative and macroscopic random matrix theories give a good account of the phenomena. In the latter case [2], one assumes that the total transfer matrix can be built up multiplicatively using transfer matrices chosen from a simple statistical ensemble with only symmetry constraints applied. The success of this theory is perhaps the best theoretical demonstration of the ubiquity of mesoscopic interference effects.

More recently, interest has focused on quantum transport in ballistic microstructures—structures in which impurity scattering can be neglected so that only scattering from the boundaries of the conducting region is important [1]. Quantum interference effects in such structures [3–9] depend on whether the classical dynamics is regular or chaotic [10]. Recent experiments [11] have detected a difference between the transport properties of nominally regular and chaotic structures.

The theoretical work on this subject [3–7] has concentrated on either numerical quantum calculations or semiclassical theory. On the other hand, it has been proposed [8,9] that chaotic scattering in the quantum regime [10] should be described by a random matrix theory for the S matrix. The emphasis in both that work and recent work on the S matrix of disordered structures [12] is on the eigenphases of S. The eigenphases, however, are not directly connected to transport because they involve both reflection and transmission. In contrast, we derive the implications of a random S-matrix theory for the quantum transport properties and provide numerical evidence that this theory applies to the class of ballistic microstructures investigated experimentally. In this way we obtain experimentally accessible predictions for the quantum transport properties of chaotic billiards.

A quantum scattering problem is described by its S matrix. For scattering involving two leads (see Fig. 1) each with N channels and width W, we have

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \tag{1}$$

where r, t are the $N \times N$ reflection and transmission matrices for particles from the left and r', t' for those from the right. In terms of S, the conductance for spinless particles is [1]

$$G = (e^2/h)T = (e^2/h)\mathrm{Tr}\{tt^\dagger\}. \tag{2}$$

Current conservation implies S is unitary, $SS^\dagger = I$, and for time-reversal symmetry ($B = 0$) S is symmetric.

We concentrate on situations where the statistics of the scattering can be described by assigning to S an "equal a

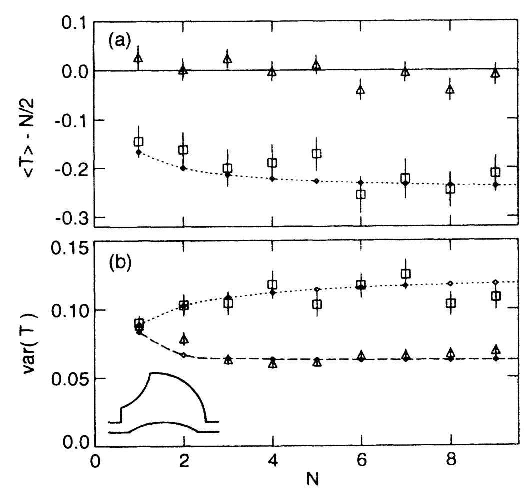

FIG. 1. The magnitude of the (a) weak-localization correction and (b) conductance fluctuations as a function of the number of modes in the leads, N. The numerical results for $B = 0$ (squares with statistical error bars) agree with the prediction of the COE (dotted line), while those for $B \neq 0$ (triangles) agree with the CUE (dashed line). The inset shows a typical cavity. The numerical results involve averaging over (1) energy at fixed N (50 points), (2) 6 different cavities obtained by changing the stoppers, and (3) 2 magnetic fields for $B \neq 0$ ($BA/\phi_0 = 2, 4$ where A is the area of the cavity).

0031-9007/94/73(1)/142(4)$06.00

priori distribution" once the symmetry restrictions have been imposed. In particular, the possibility of "direct" processes, caused, for instance, by short trajectories and giving rise to a nonvanishing averaged S matrix [8,13], is ruled out. The appropriate ensembles are well known in classical random matrix theory [14] and are called the circular orthogonal ensemble (COE, $\beta = 1$) in the presence of time-reversal symmetry and the circular unitary ensemble (CUE, $\beta = 2$) in its absence. These ensembles are defined through their invariant measure: the measure on the matrix space which is invariant under the appropriate symmetry operations. To be precise [14], $d\mu(S) = d\mu(S')$, where $S' = U_0 S V_0$, and U_0, V_0 are arbitrary fixed unitary matrices in the case of the CUE with the restriction $V_0 = U_0^T$ in the COE. Numerical evidence for the validity of this random matrix theory for describing quantum-chaotic scattering can be found in Ref. [8].

Perhaps the most widely studied mesoscopic transport effects are the magnitude of the conductance fluctuations—how much the conductance varies as a magnetic field or gate voltage is applied—and the size of the weak-localization correction to the average conductance at $B = 0$ [1]. We therefore start by deriving $\langle T\rangle$ and $\mathrm{var}(T)$ where we use an integration over the invariant measure as the average. Such integrals have been evaluated previously [15], and we find that

$$\int d\mu(S)|t_{ab}|^2 = \frac{1}{2N + \delta_{1\beta}},$$

$$\int d\mu(S)|t_{ab}|^2|t_{cd}|^2 = \frac{2(N+\delta_{1\beta})(1+\delta_{ac}\delta_{bd}) - \delta_{ac} - \delta_{bd}}{2N(2N+1)(2N-1+4\delta_{1\beta})}.$$

Performing the trace over channels in Eq. (2), we obtain

$$\langle T\rangle - N/2 = -\delta_{1\beta}N/(4N+2) \to (-1/4)\delta_{1\beta}, \tag{3a}$$

$$\mathrm{var}(T) = \begin{cases} \dfrac{N(N+1)^2}{(2N+1)^2(2N+3)} \to \dfrac{1}{8}\,, & \text{COE}\,, \\ \dfrac{N^2}{4(4N^2-1)} \to \dfrac{1}{16}\,, & \text{CUE}\,, \end{cases} \tag{3b}$$

where the limit is as $N \to \infty$.

We make several comments concerning these results. (1) Previously, semiclassical theory and numerical calculations suggested that the weak-localization correction, $\langle T\rangle - N/2$, and the magnitude of the conductance fluctuations, $\mathrm{var}(T)$, are independent of the size of the system for chaotic billiards [3]. This is the analog of the "universality" of the conductance fluctuations in the diffusive regime [1]. *Since the number of modes is proportional to the size of the system ($N = \mathrm{int}[kW/\pi]$), our $N \to \infty$ results show that the conductance fluctuations and weak localization are universal within the random S-matrix theory.* (2) In the large N limit, $\mathrm{var}(T)$ in the presence of time-reversal symmetry is twice as large as in its absence, as in the diffusive regime, demonstrating the universal effect of symmetry. (3) Both quantities show some variation in the small N regime typical of the experiments [11]; for instance, the ratio of $\mathrm{var}(T)$ in the presence and absence of symmetry is not 2. (4) The values obtained in the $N \to \infty$ limit are the same as those from an equivalent random matrix theory for the Hamiltonian in which the billiard is described by the Gaussian ensembles and the conductance follows from coupling the billiard to leads in a random way [9].

The predictions of the random matrix theory are compared to the conductance of a stadium billiard in Fig. 1 computed using the method of Ref. [16]. In these calculations, the S matrix varies as a function of energy because of the cavity resonances. We estimate that the resonances are moderately overlapping for $N = 1$ and that the width to spacing ratio increases linearly with N. The basic assumption (*ergodic hypothesis*) is that through these fluctuations S covers the matrix space with uniform probability. This should apply to billiards in which the effect of short nonchaotic paths is minimized. We therefore use a stadium billiard in which (1) a stopper blocks any direct transmission between the leads, (2) a stopper blocks the whispering gallery trajectories which hug the half-circle part of the stadium, and (3) the stadium is asymmetrized to break all reflection symmetries. We obtain excellent agreement between the energy averages found numerically and the invariant-measure ensemble averages introduced above. In particular, both the variation at small N and the ratio of $\mathrm{var}(T)$ in the presence and absence of time-reversal symmetry are verified.

Motivated by this good agreement, we consider more detailed predictions of the random S-matrix theory: we derive the full distribution of T for small N and the statistics of the eigenvalues of $tt^\dagger$, denoted $\{\tau\}$. We obtain these results by expressing the invariant measure in terms of a set of variables that includes the $\{\tau\}$. Any unitary matrix of the form in Eq. (1) can be written as [17]

$$S = \begin{bmatrix} v^{(1)} & 0 \\ 0 & v^{(2)} \end{bmatrix} \begin{bmatrix} -\sqrt{1-\tau} & \sqrt{\tau} \\ \sqrt{\tau} & \sqrt{1-\tau} \end{bmatrix} \begin{bmatrix} v^{(3)} & 0 \\ 0 & v^{(4)} \end{bmatrix}, \tag{4}$$

where τ is the $N \times N$ diagonal matrix of the $\{\tau\}$ and the $v^{(i)}$ are arbitrary unitary matrices except that $v^{(3)} = (v^{(1)})^T$ and $v^{(4)} = (v^{(2)})^T$ in the presence of time-reversal symmetry. It is a general property of measures on vector spaces [18] that a differential arc length written in the form $d\sigma^2 = \sum_{ab} g_{ab}dx^a dx^b$ implies that the volume measure is $d\mu(V) = \sqrt{\det(g)}\prod_a dx^a$. In our case the differential arc length is simply $d\sigma^2 = \mathrm{Tr}\{dS^\dagger dS\}$. Substituting for S the form in Eq. (4), one finds ($\beta = 1, 2$)

$$d\mu(S) = P_\beta(\{\tau\})\prod_a d\tau_a \prod_i d\mu(v^{(i)}), \tag{5}$$

where the joint probability distribution of the $\{\tau\}$ is

$$P_2(\{\tau\}) = C_2 \prod_{a<b} |\tau_a - \tau_b|^2, \tag{6a}$$

$$P_1(\{\tau\}) = C_1 \prod_{a<b} |\tau_a - \tau_b| \prod_c 1/\sqrt{\tau_c}\,, \tag{6b}$$

$d\mu(v^{(i)})$ is the invariant (Haar's) measure on the unitary group [19], and C_β are normalization constants [20].

The distribution of $T = \sum_{a=1}^{N} \tau_a$ follows by integration over the joint probability distribution. This can be carried out for small N; for instance, in the trivial case $N = 1$, $w(T) = 1$ for the CUE and $w(T) = 1/2\sqrt{T}$ for the COE. For $N = 1-3$ the $w(T)$ derived analytically from the random matrix theory are plotted in Fig. 2 and compared to numerical data for billiards. *Note the dramatic difference between the CUE and COE in the single mode case, and the difference within each ensemble between the $N = 1$ and $N = 2$ cases.* The results for $N = 3$ are close to a Gaussian distribution.

The agreement between the numerical and theoretical results in Fig. 2 is very good in terms of the dependence on both B and N [21]. These effects should be observable in experiments, for which N is typically small, and would provide a clear test of the applicability of random S-matrix theory to experimental microstructures.

Though not experimentally accessible, the $\{\tau\}$ are theoretically interesting because of their fundamental relation to the conductance. We obtain further information by writing the joint probability density in the form

$$P_\beta(\{\tau\}) = C_\beta \exp\left\{-\beta\left[\sum_{a<b} \ln|\tau_a - \tau_b| + \sum_c V_\beta(\tau_c)\right]\right\}, \tag{7}$$

with $V_2(\tau) = 0$ and $V_1(\tau) = \frac{1}{2}\ln\tau$. This is exactly the form of the joint density in the global-maximum-entropy approach to transport in disordered systems [2] and in the Gaussian ensembles [14]. Many statistics of such distributions are known asymptotically as $N \to \infty$ [14,22–25]. For instance, the known form of the asymptotic two-point correlation function [22,25] can be used to obtain

$$\mathrm{var}(T) = \int_0^1 d\tau \int_0^1 d\tau' \tau\tau' \rho_2(\tau,\tau') \to 1/8\beta. \tag{8}$$

This result agrees with Eq. (3b) and is independent of the potential $V(\tau)$. Therefore the asymptotic value of $\mathrm{var}(T)$ is the same for a large class of random matrix theories, a stronger form of "universality" [24].

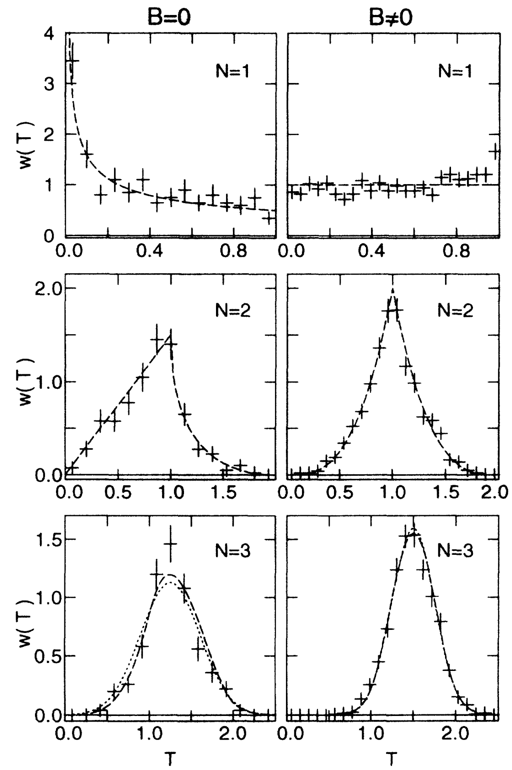

FIG. 2. The distribution of the transmission intensity at fixed $N = 1$, 2, or 3 in both the absence (first column) and presence (second column) of a magnetic field. The numerical results (plusses with statistical error bars) are in good agreement with the predictions of the circular ensembles (dashed lines). Note the striking difference between the $N = 1$ and $N = 2$ results and between the $B = 0$ and $B \neq 0$ results for $N = 1$. For $N = 3$ the distribution approaches a Gaussian (dotted lines). The cavities used are the same as those in Fig. 1; for $B \neq 0$, $BA/\phi_0 = 2$, 3, 4, and 5 were used.

In the CUE, the form of the joint density in Eq. (7) is suitable for the random matrix theory method of orthogonal polynomials [2,14]. Because $V = 0$ and the $\{\tau\}$ are restricted to $[0,1]$ the Legendre polynomials are appropriate [26]. In terms of these polynomials, the exact eigenvalue and two-point correlation function are [22,26]

$$\rho(\tau) = \frac{N^2}{4\tau(1-\tau)}[P_N^2(\alpha) - 2\alpha P_N(\alpha)P_{N-1}(\alpha) + P_{N-1}^2(\alpha)],$$

$$\rho_2(\tau,\tau') = \rho(\tau)\delta(\tau-\tau')/N - [P_N(\alpha)P_{N-1}(\alpha') - P_{N-1}(\alpha)P_N(\alpha')]^2/16(\tau-\tau')^2, \tag{9}$$

where $\alpha \equiv 2\tau - 1$. Using the asymptotic expansion of the Legendre polynomials as $N \to \infty$ and some smoothing, one finds that $\rho(\tau) \to N/\pi\sqrt{\tau(1-\tau)}$ [26] and recovers the expression in Refs. [22,25] for the asymptotic two-point correlation function. Previous work has shown that the statistics of the eigenvalues $\{\tau\}$ follows that of the Gaussian unitary ensemble in the large N limit [26].

In summary, we have derived the consequences for quantum transport on the assumption that the S matrix of a chaotic cavity follows the circular ensembles. We have shown that the magnitude of both the conductance fluctuations and the weak localization is universal in the large N limit, at least within the class of systems considered here [27]. The small N limit is most relevant experimentally, and here we find a striking dependence of the full distribution of T on both N and magnetic field (Fig. 2). In closing, we emphasize that we have neglected the "direct" scattering due to short paths ($\langle S \rangle = 0$); since such scattering is important in many chaotic cavities, the effect of these processes remains an important open question, which, in principle, could be investigated using the information-theoretic model of Ref. [13].

We thank R. A. Jalabert for several valuable discussions. One of us (P.A.M.) appreciates the kind hospitality of the Wissenschaftkolleg zu Berlin where part of this work was performed.

Note added.—While writing this paper we received a preprint by R. A. Jalabert, C. W. J. Beenakker, and J.-L. Pichard which contains some similar material. We thank J.-L. Pichard for sending us this preprint.

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[20] In Eq. (5) we have omitted phase factors which reflect the arbitrariness of the decomposition Eq. (4) for the CUE case. None of the results reported here is affected; see Ref. [17] for a full discussion. Also note that Π_i runs from $i = 1$ to 2 for the COE and $i = 1$ to 4 for the CUE.

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[27] We have considered structures with two equal size leads and $\langle S \rangle = 0$. It is straightforward to consider leads of different size: suppose there are N_1 (N_2) propagating modes in lead 1 (2). Then, for example, for $N_1, N_2 \to \infty$ with $N_1/N_2 \equiv K$, $\langle T \rangle_{\text{COE}} - \langle T \rangle_{\text{CUE}} = -K/(1+K)^2$, $\text{var}(T)_{\text{COE}} = 2\text{var}(T)_{\text{CUE}} = 2K^2/(1+K)^4$, and we have universality classes labeled by K. The weak dependence on K limits the "universality" of our results.